Black holes as probes of Lorenz Invariance Violation

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Cankarjev Dom, Ljubljana, Slovenia

$$\mathcal{H}^b = D_a\left(\tilde{\pi}^{ab}\right) \approx 0 \quad (11)$$

$$\gamma A^i_a := \Gamma^i_a + \gamma K^i_a$$

$$S = \frac{\gamma \tilde{E}^a_i \tilde{E}^b_j}{\sqrt{\det(\tilde{E})}}\left[\epsilon^{ij}_{\ \ k} F^k_{ab} - 2(1+\gamma^2) K^i_{[a} K^j_{b]}\right] \approx 0$$

$$\sqrt{q}\left[R^{(3)} + q^{-1}\left(\frac{1}{2}\tilde{\pi}^2 - \tilde{\pi}^{ab}\ldots\right)\right] \quad (12)(10)$$

phase : $v_{\text{ph}} = \frac{\omega}{|\vec{k}|} \equiv \frac{c}{n(\omega)} = c$ (1)

group : $v_{\text{gr}} = \frac{\partial \omega}{\partial |\vec{k}|} \equiv \frac{c}{n_{\text{gr}}(\omega)} = c\,,$ (2)

front : $v_{\text{front}} = c/n(\infty) = c$ (3)

and

$$n_{\text{gr}}(\omega) = n(\omega) + \omega\frac{\partial n(\omega)}{\partial \omega}$$ (4)

$$\mathcal{V}^{short}_{D0-D8} = \int_{r_{\min}}^{\ell_s} n^\star(r)\,\frac{\pi\alpha'}{12}\frac{v^2}{r^3}\,dr + \int_{-r_{\min}}^{-\ell_s} n^\star(r)\,\frac{\pi\alpha'}{12}\frac{v^2}{r^3}\,dr \quad \text{for } r \ll \sqrt{\alpha'}$$

$$\mathcal{V}^{long}_{D0-D8} = n^\star(r)\frac{v^2}{8\pi\alpha'}\,dr \quad \text{for } r \gg \sqrt{\alpha'}$$

$$\rho^{D8-D0}_{\text{total}} = \int_{-\ell_s}^{-R_1(t)} n^\star(r)\frac{v^2}{8\pi\alpha'}\,dr + \rho_0$$

$$\rho^{D8-D0}_{\text{total}} \simeq -n_{\text{local}}\frac{\pi}{12}v(1-\alpha')\left(R_1(t)^2 - \alpha'\right) + n_{\text{left}}\frac{1}{16\pi\alpha'}\ldots$$

$$n_{\text{right}}v^2\frac{1}{16\pi\alpha'}\left(R_2(t)^2 - \alpha'\right) + \rho_0$$ (5)

$$\frac{\Omega'_\chi h_0^2}{(\Omega_\chi h_0^2)_{\text{no source}}} \simeq \left[1 + 207.38 g_s^2\frac{m^2}{M_s^2}x_f^{-2}\left(\sum_{i=1}^3 \Delta_i^2\right)\right]^{1/2}\left[1 + g_s^2\frac{m^2}{M_s^2}\left(\sum_{i=1}^3\Delta_i^2\right)\left(1+6x_0^{-1}\right)\right]$$

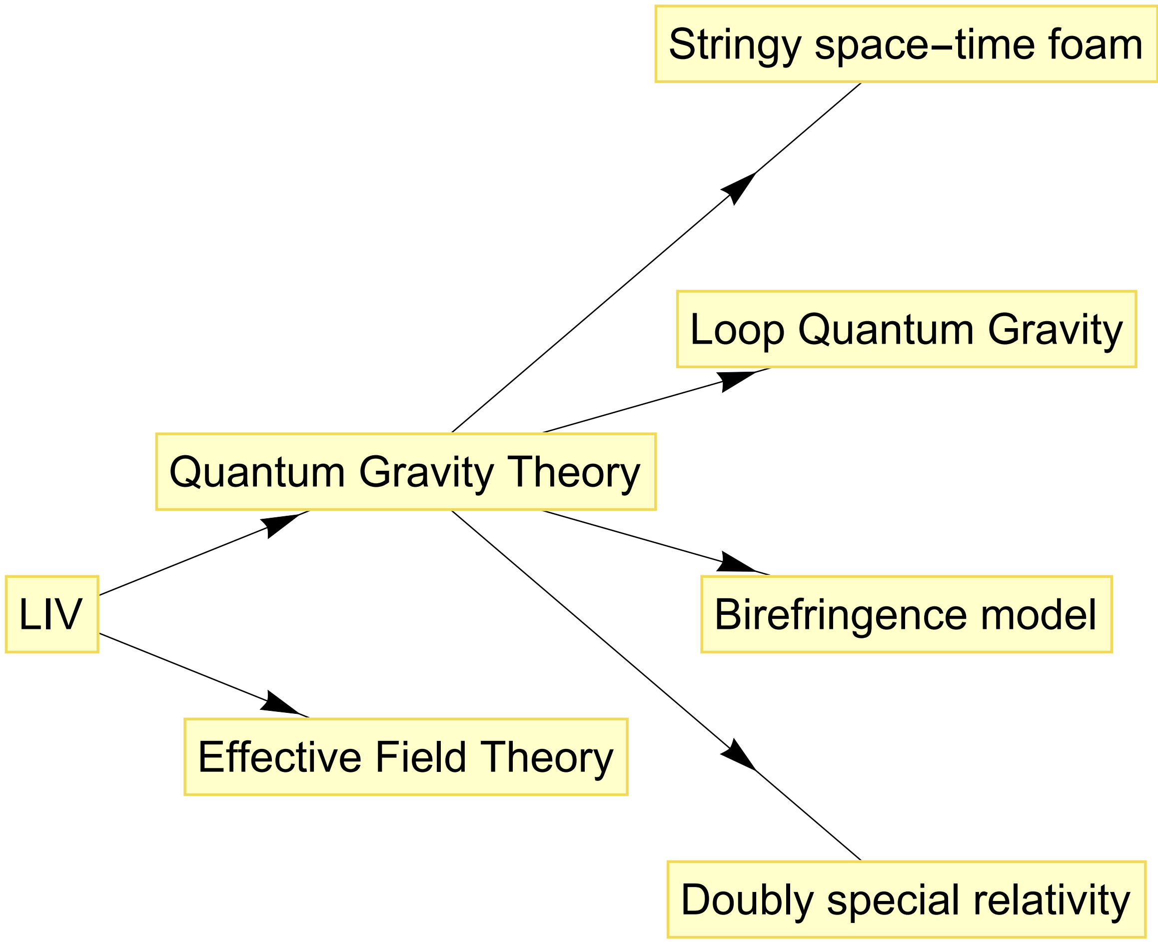
“Detection of time-delays in VHEs from BHs”

Overview

- LIV and Quantum Gravity
- Experimental searches
- Black holes as probes
- Observations & Results

LIV theories

Theoretically numerous models



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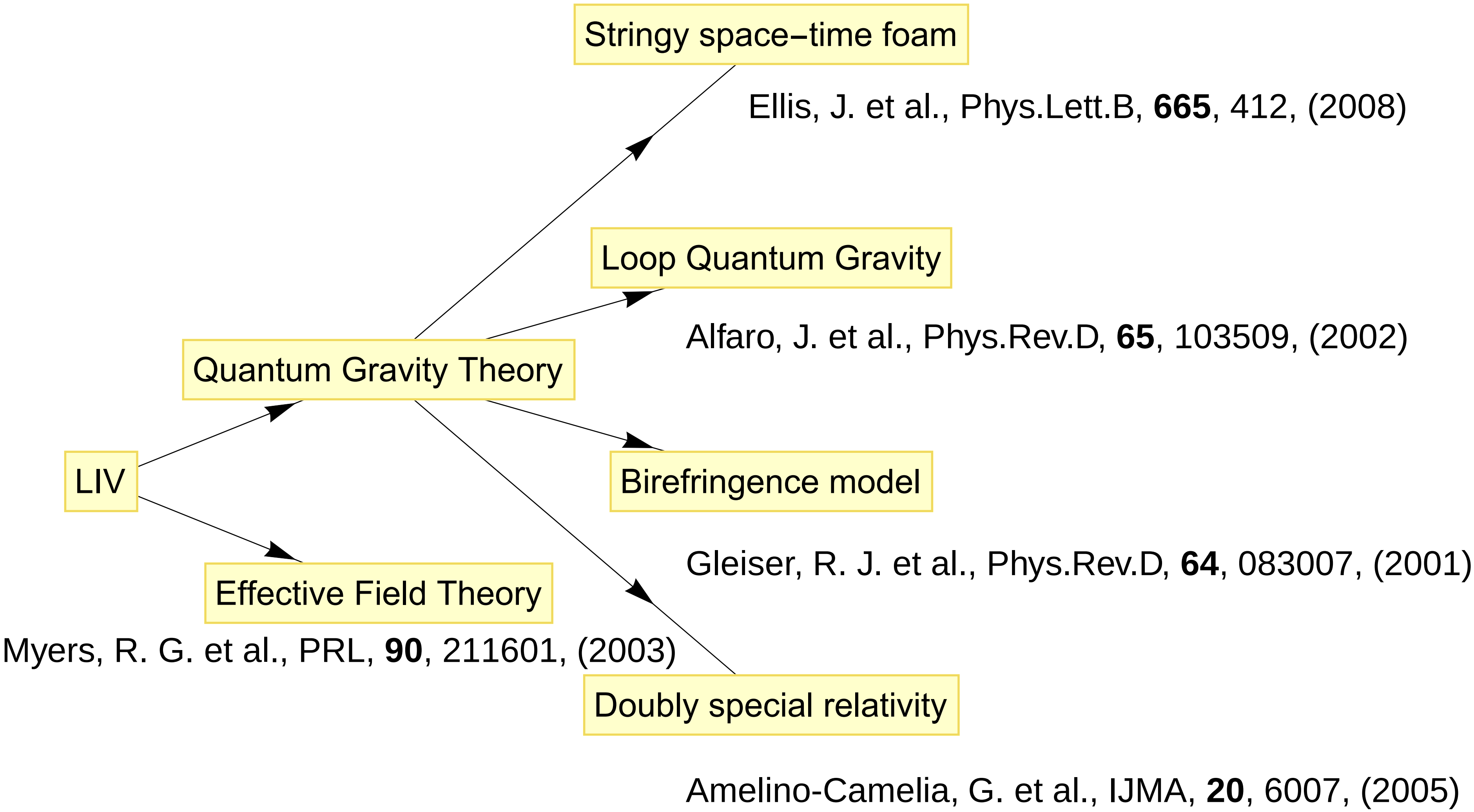

LIV theories

Theoretically numerous models

All of them very difficult to prove wrong on a theoretical basis!

What Quantum Gravity really is?

- There is no consistent coupling between a classical and a quantum system

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- There is no consistent coupling between a classical and a quantum system
- Description of the gravity force in terms of quantum mechanics
- Quantum behaviour of gravity not TOE (e.g. stringy)

Light Propagation

"Light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body."

Einstein.A, Ann.Phys., **322**, 891, (1905)

Light Propagation

"Light is always propagated in empty space with a definite velocity c_0 which is independent of the state of motion of the emitting body."

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Light Propagation

Dispersion: The phase velocity of a wave depends on its frequency i.e. energy E.

$$u(E) = c_0 \times n(E)^{-1}$$

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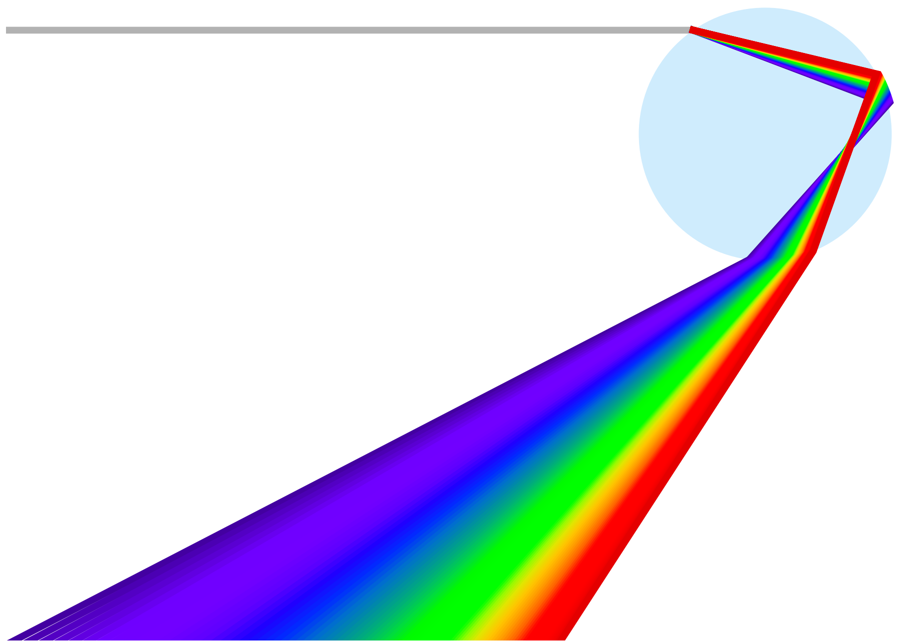

Light Propagation

$$c(E) = c_0 \times \left[1 + \xi \frac{E}{E_{\mathrm{P}}} + \zeta \left(\frac{E}{E_{\mathrm{P}}} \right)^2 + \mathcal{O} \left(\frac{E}{E_{\mathrm{P}}} \right)^3 \right]$$

Amelino-Camelia, G. et al., Nature, **393**, 763, (1998)

Light Propagation

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Scales:

- Energies

$$E_\mathrm{P} = \sqrt{\frac{\hbar c^5}{G}} \simeq 1.22 \times 10^{19} \;\; \mathrm{GeV}$$

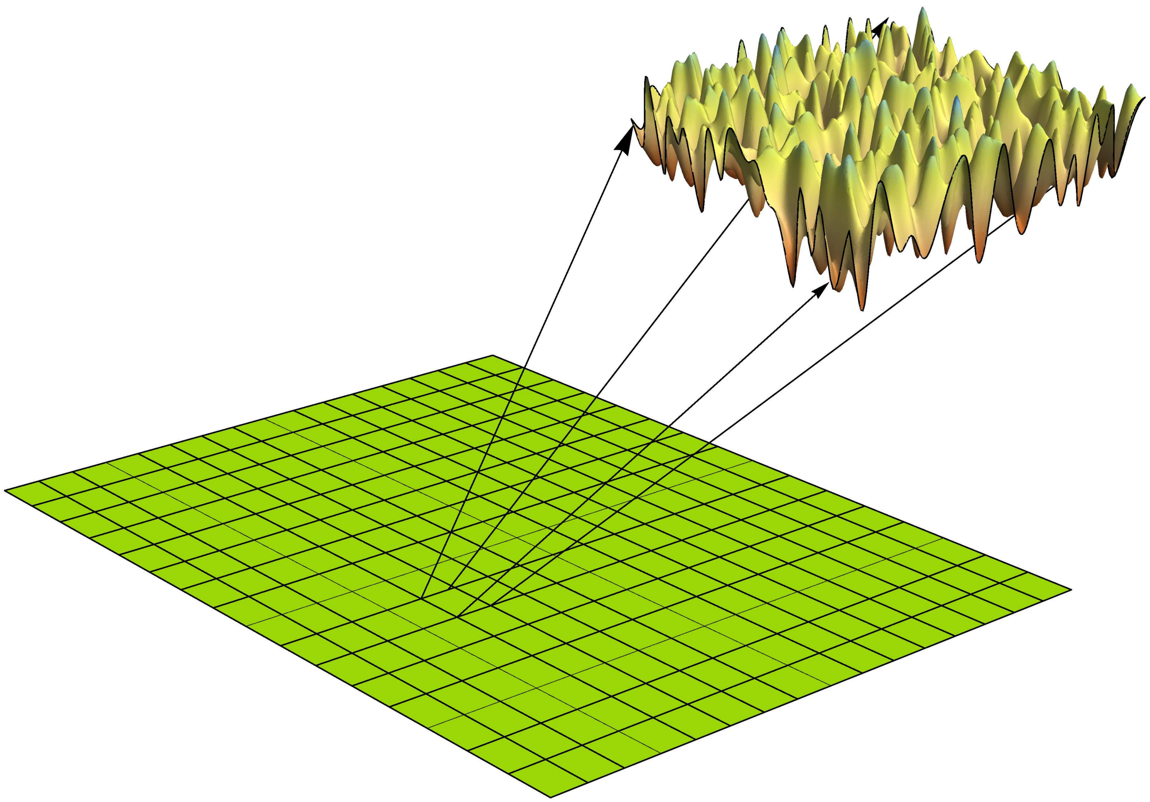
- Lengths

$$l_\mathrm{P} = \sqrt{\frac{\hbar G}{c^3}} \simeq 1.61 \times 10^{-33} \;\; \mathrm{cm}$$

- Times

$$t_\mathrm{P} = \sqrt{\frac{\hbar G}{c^5}} \simeq 5.39 \times 10^{-44} \;\; \mathrm{s}$$

Light Propagation



Experimental searches

- Collider searches

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  $10^{-18} - 10^{-19}$ m very far away from Planck scale.

- CERN

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  CMB polarization modes

  Tsujikawa et al., Classical and Quantum Gravity, **21**, 5767

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• Planck

• Atacama B-mode Search

• Background Imaging of Cosmic Extragalactic Polarization

• QUEST at DASI

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- Planck
- Atacama B-mode Search (ABS)
- Background Imaging of Cosmic Extragalactic Polarization (BICEP)
- QUEST at DASI (QUAD)

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$$c(E) = c_0 \left[1 + \xi \frac{E}{E_{\mathrm{P}}} + \zeta \left(\frac{E}{E_{\mathrm{P}}} \right)^2 + \mathcal{O} \left(\frac{E}{E_{\mathrm{P}}} \right)^3 \right]$$

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$$\Delta t \approx \xi \left(\frac{\Delta E}{E_{\rm P}} \right) \frac{L}{c_0}$$

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$$\frac{\Delta t}{\Delta E} \approx \frac{\xi}{\mathrm{E_P H_0}} \int_0^z \frac{(1+z')\,dz'}{\sqrt{\Omega_m(1+z')^3+\Omega_\Lambda}}$$

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$$\frac{\Delta t}{\Delta E^2} \approx \frac{3\zeta}{2\mathrm{E}_{\mathrm{P}}^2 \mathrm{H}_0} \int_0^z \frac{(1+z')^2\, dz'}{\sqrt{\Omega_m (1+z')^3 + \Omega_\Lambda}}$$

Experimental searches

- Collider searches
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Problem: Phenomena on very small length scales (10^{-33} cm) and very high energies (10^{19} GeV)

Experimental searches

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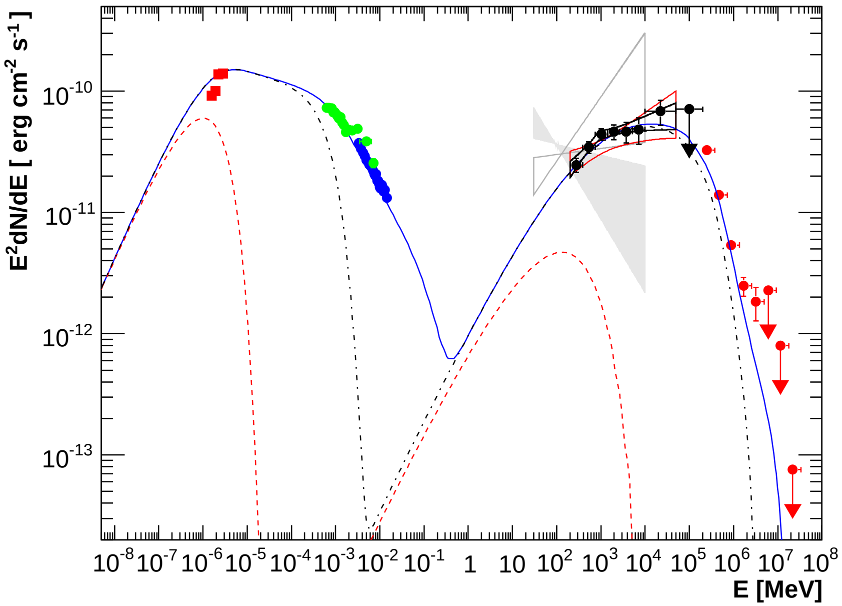
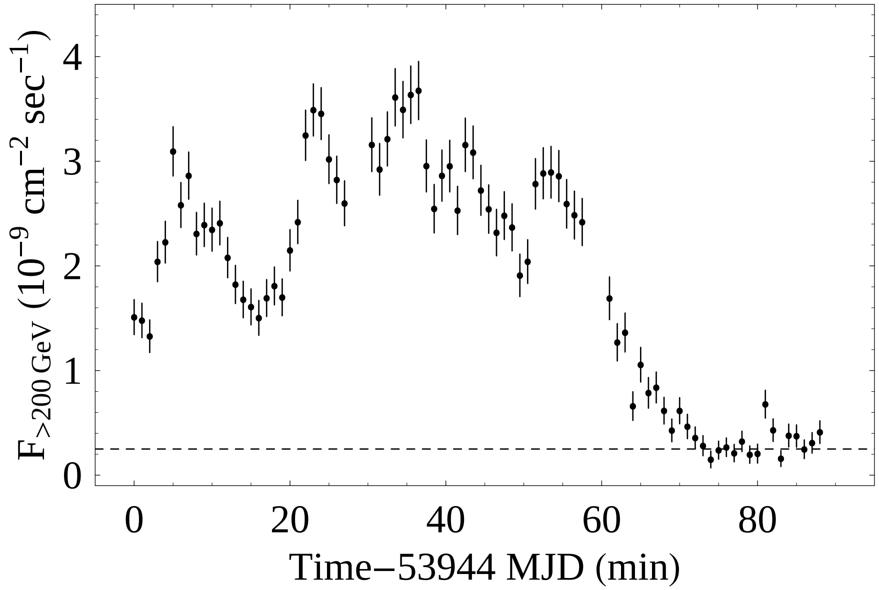
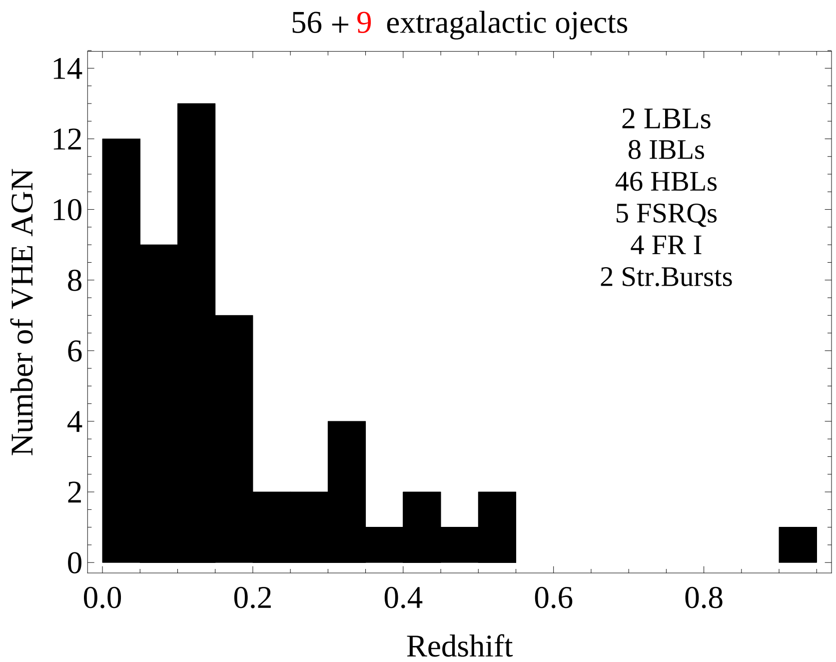
Solution: Distant VHE astrophysical sources
-Blazars-

Why to use blazars?







How do we observe them?

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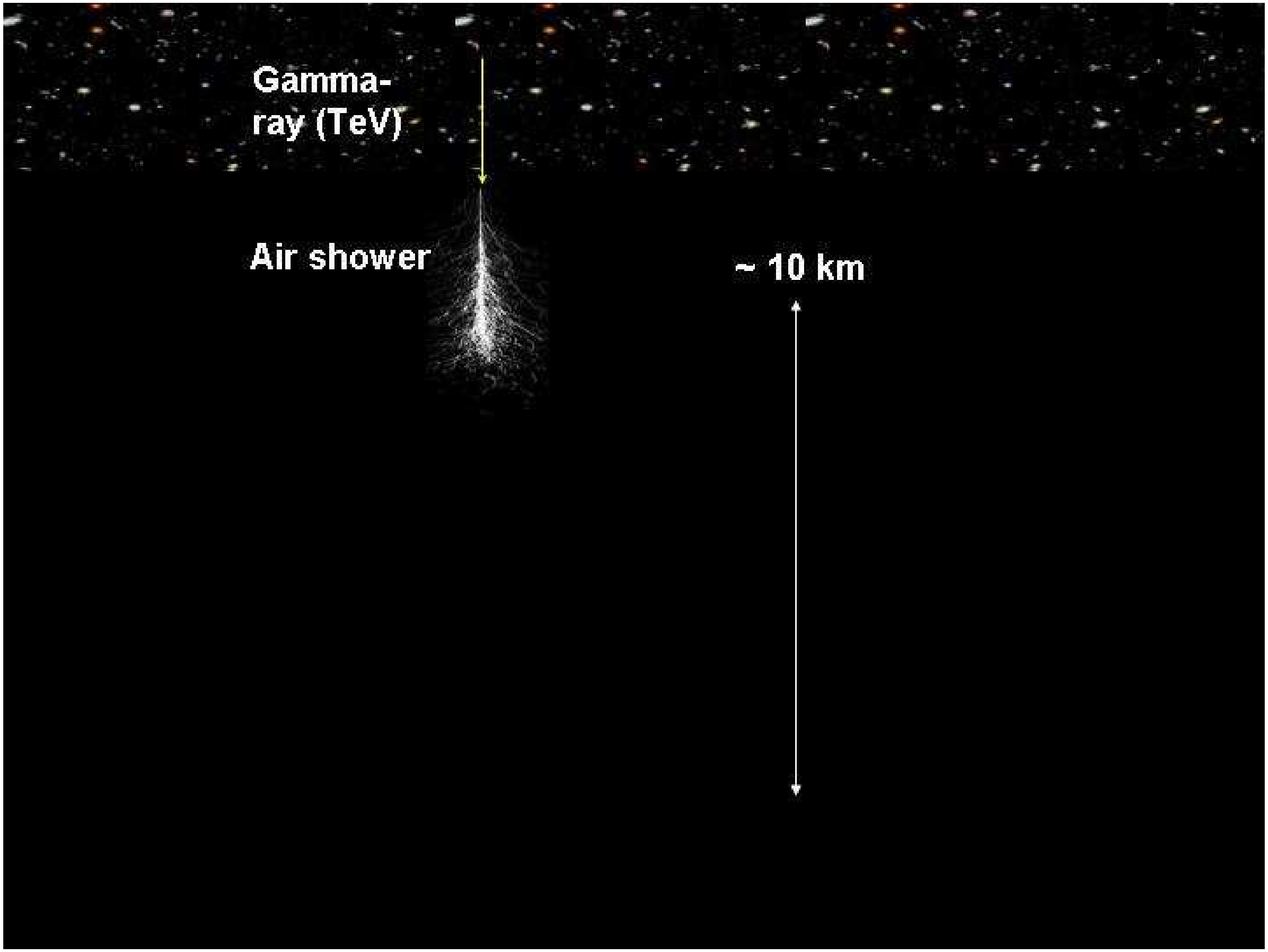

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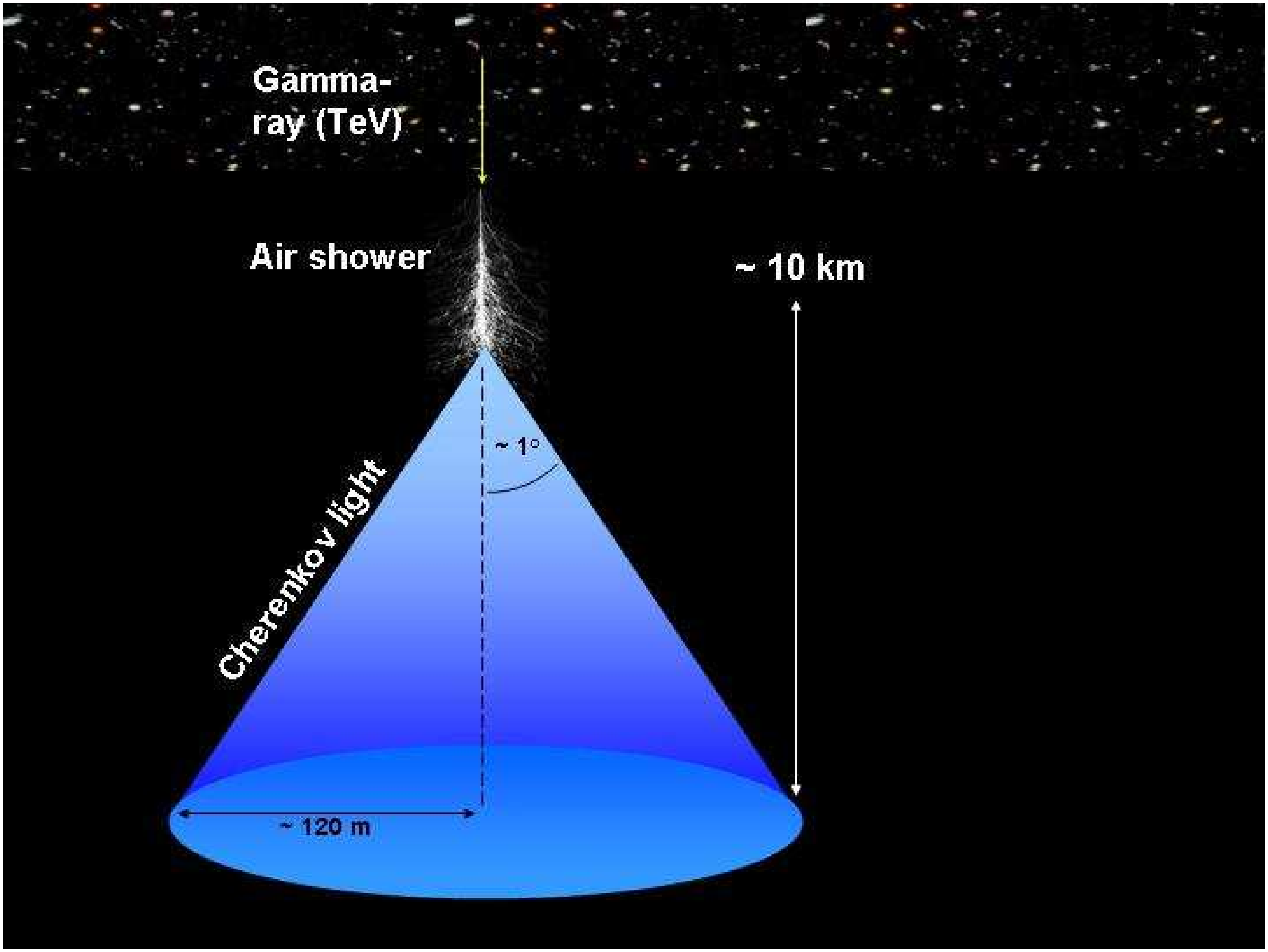

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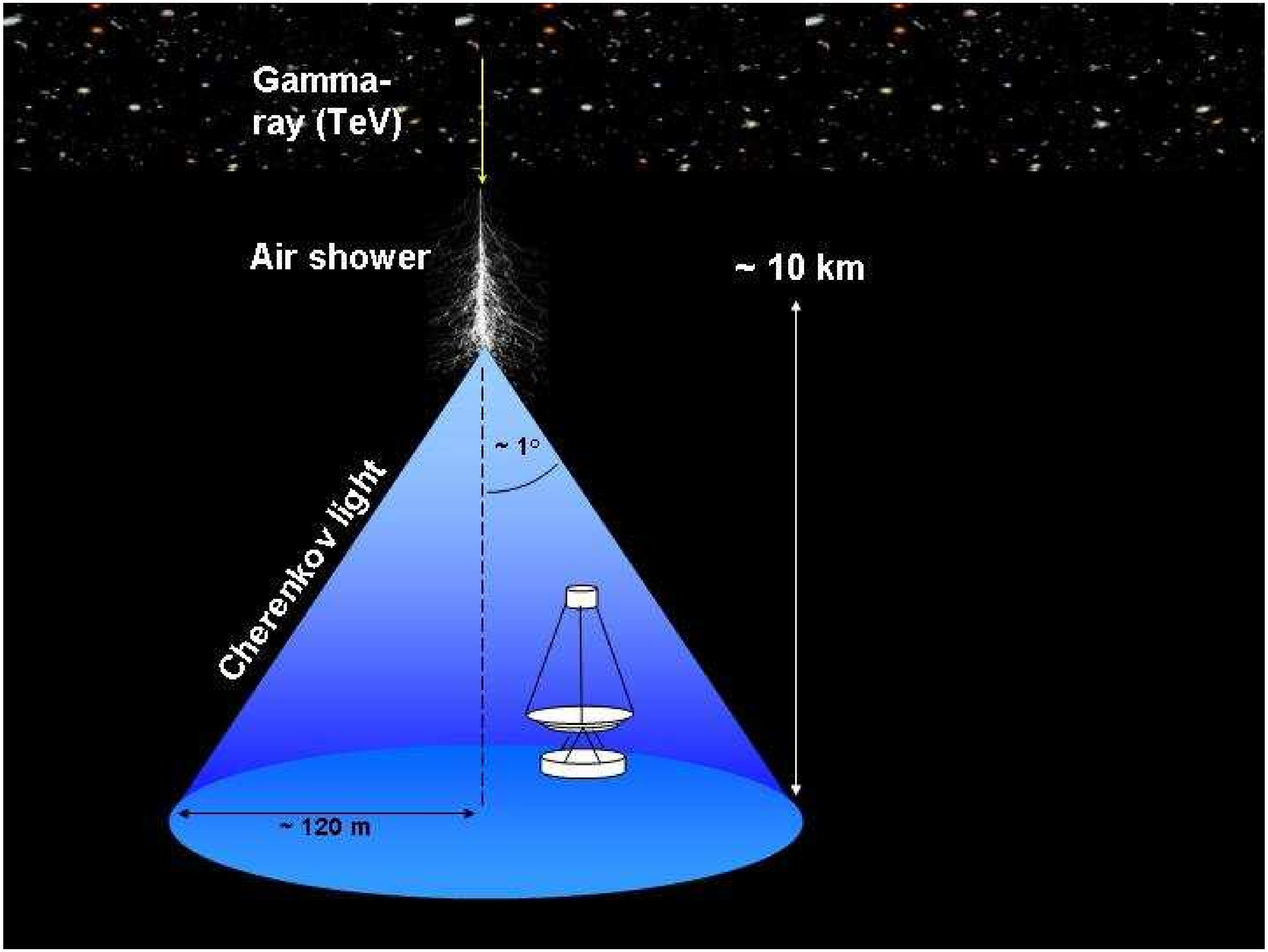

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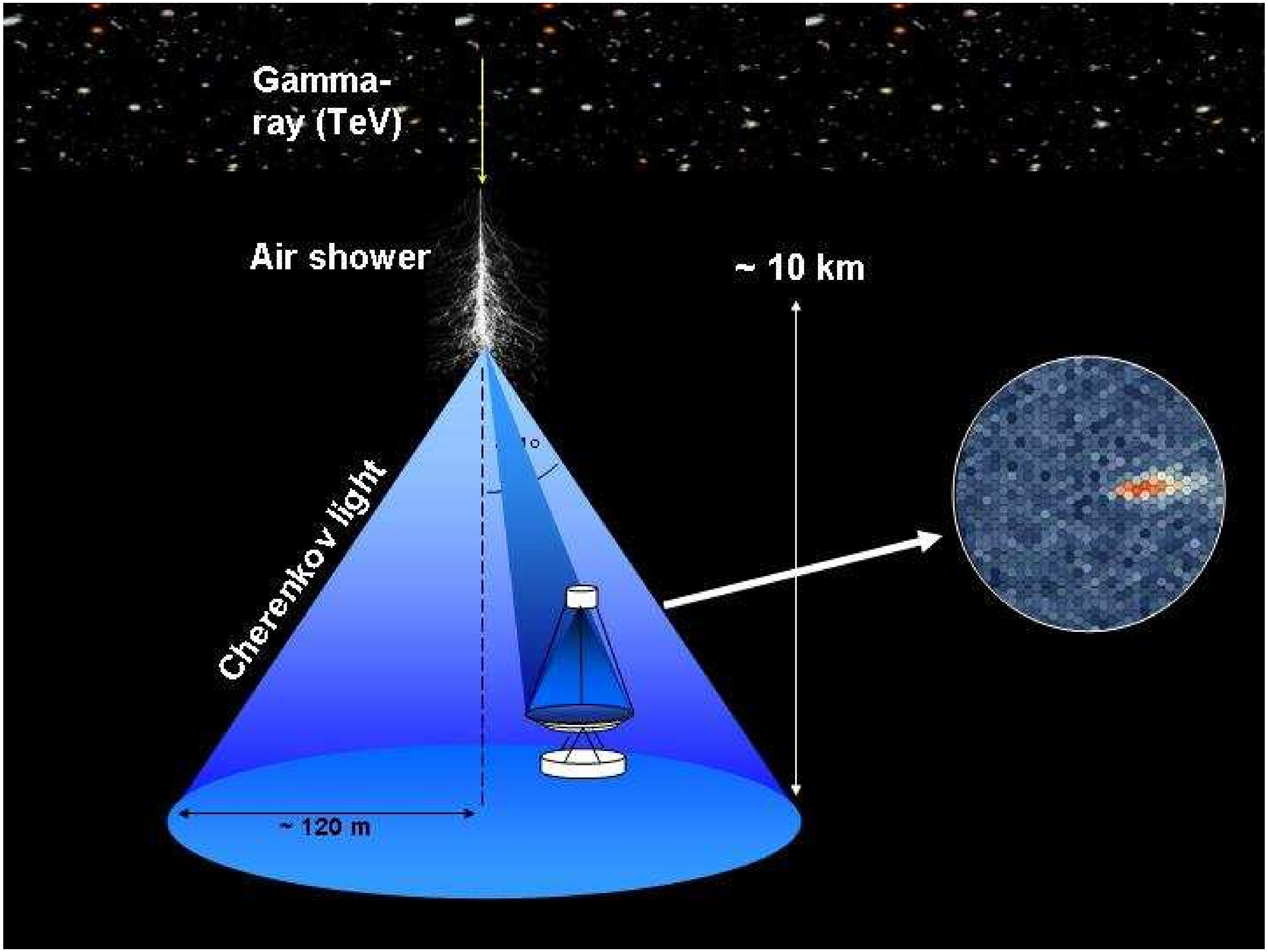

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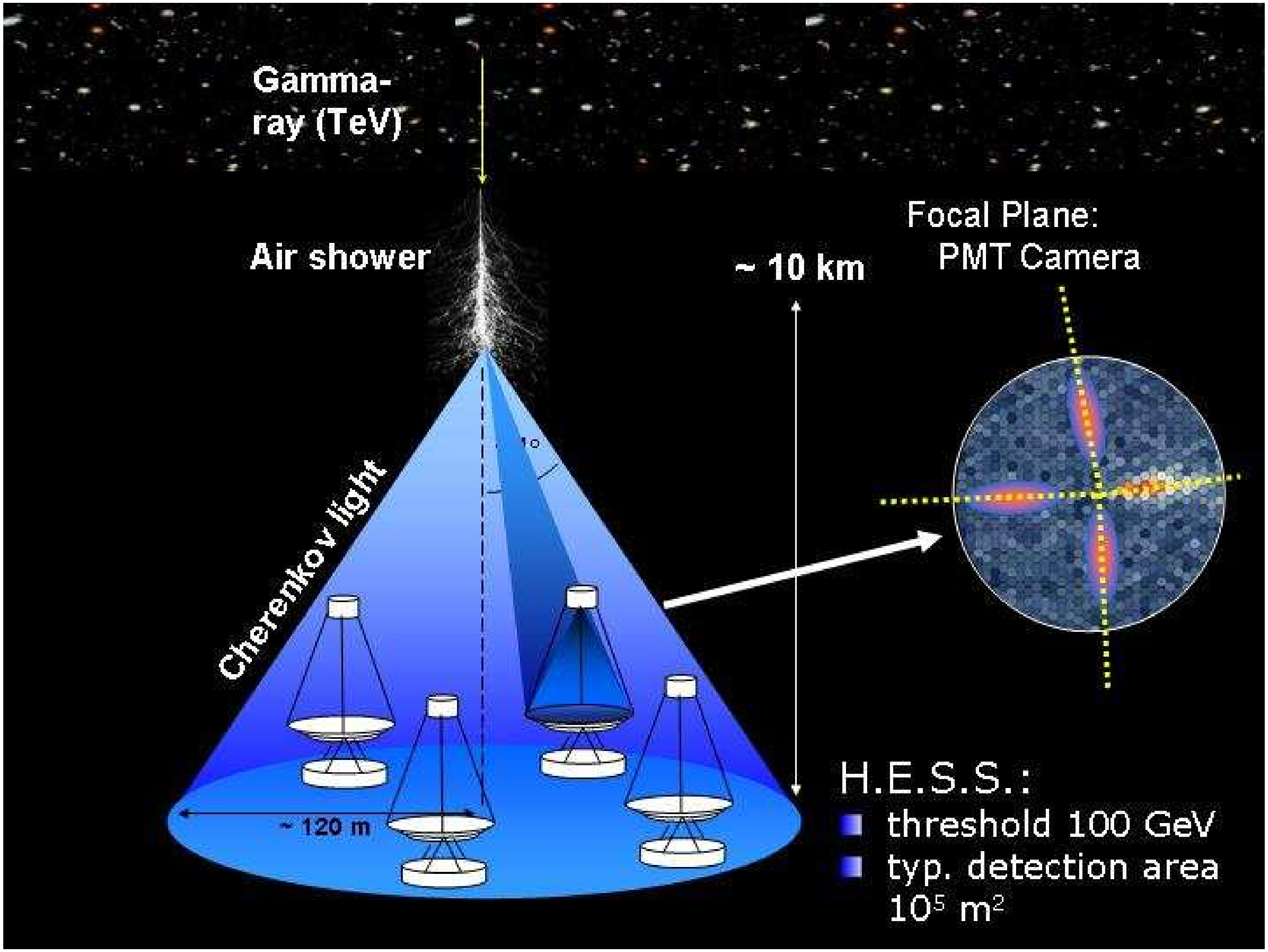
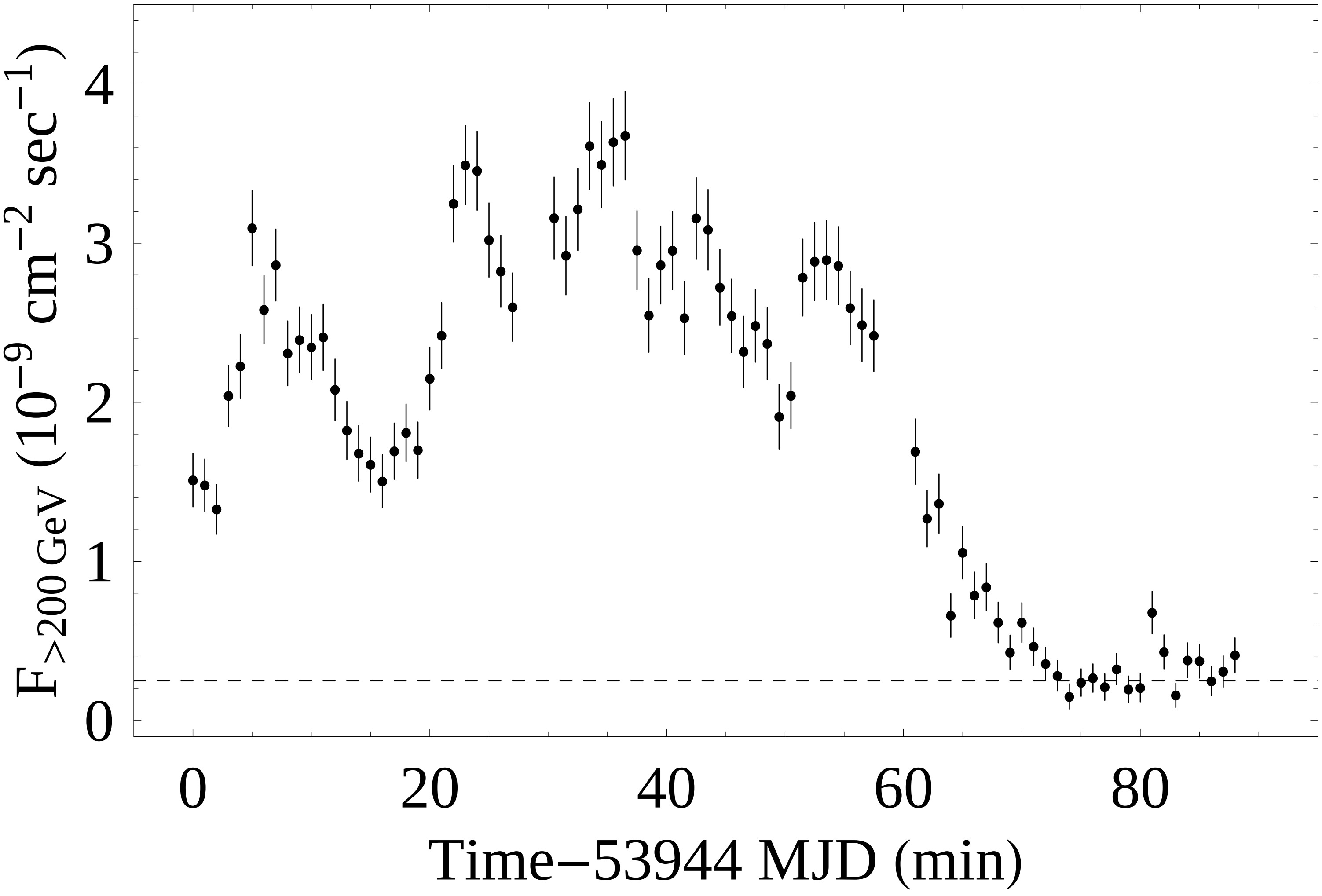


Results

PKS 2155-304 (July 2006)



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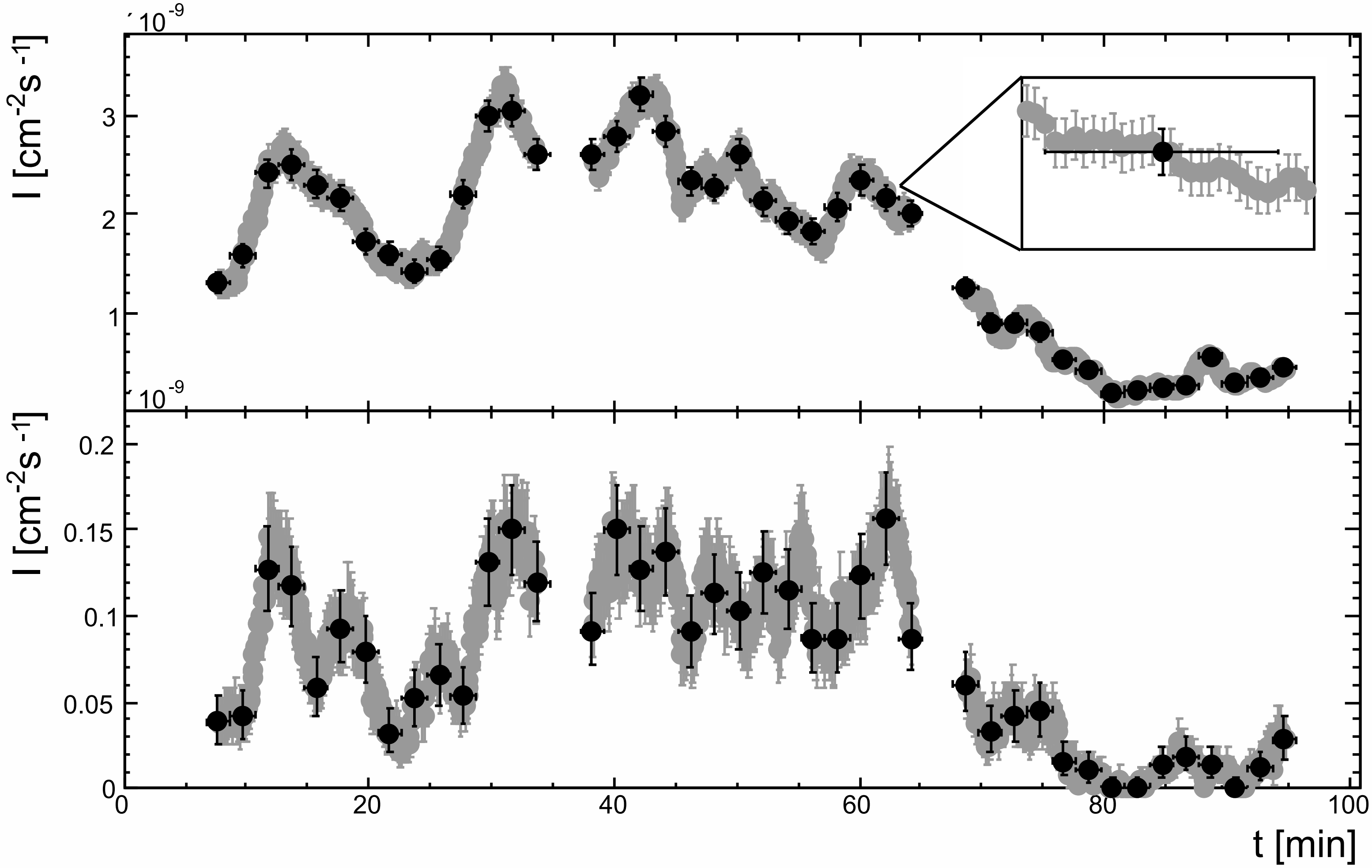
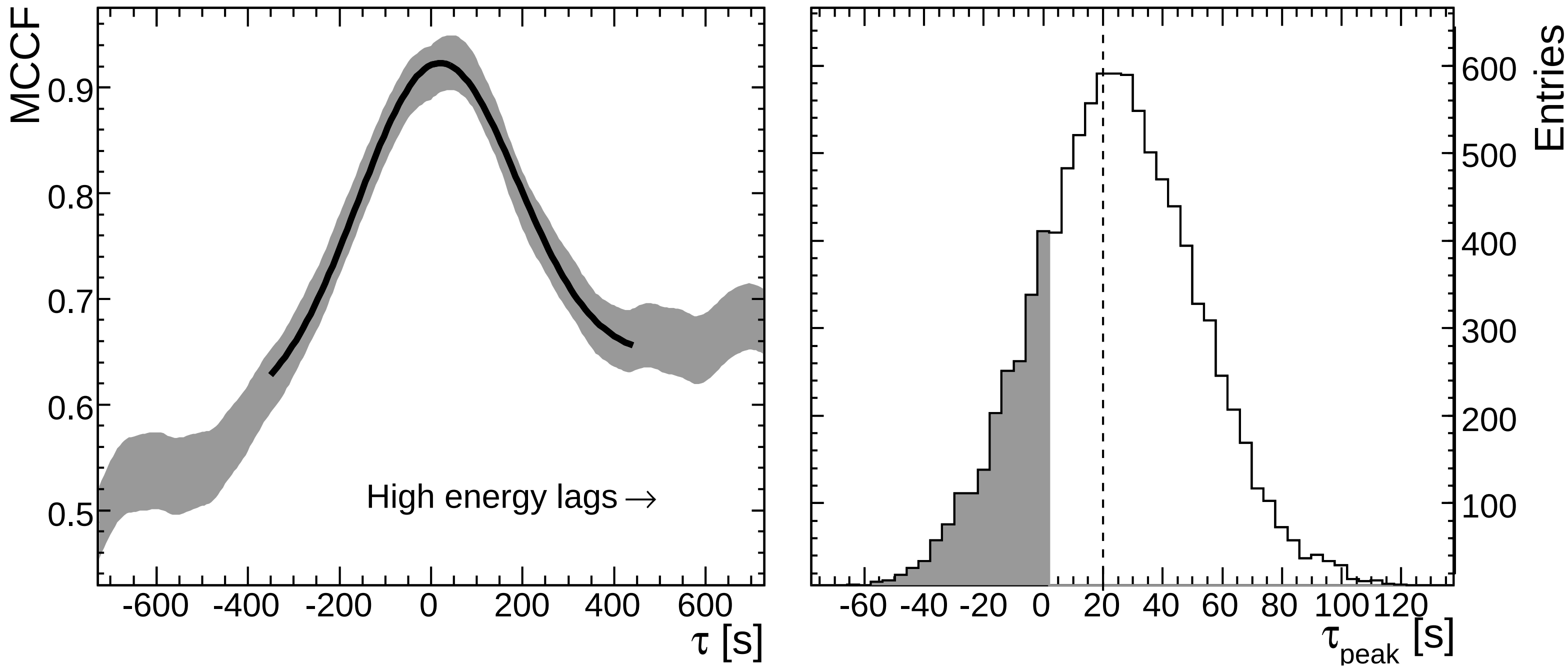


200–800 GeV

>800 GeV

Results



$$t_{\rm peak} = (20 \pm 28)\text{sec} \xrightarrow{\Delta t \approx \xi \left(\frac{\Delta E}{E_{\rm P}}\right)\frac{L}{c_0}} |\xi| < 17, \; |\zeta| < 7.3 \times 10^{19}$$

Aharonian, F. et al., (H.E.S.S. Collaboration), PRL, **101**, 170402 (2008)

Results

Methodologies: Cross-correlation, wavelets, energy cost function, maximum likelihood

- $\xi < 300$ from Mrk 421 (z=0.030)

  Biller, S. D. et al., (WHIPPLE Collaboration), PRL, **83**, 2108, (1999)

- $\xi < 60$ from Mrk 501 (z=0.034)

  Albert, J. et al., (MAGIC Collaboration), Phys. Lett. B, **668**, 253, (2008)

- $\xi < 17$ from PKS 2155-304 (z=0.117)

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- $\xi < 6$ from PKS 2155-304 (z=0.117)

  Abramowski, A. et al., (H.E.S.S. Collaboration), Astropart. Phys. **34**, 738, (2011)

What about GRBs?

FERMI

What about GRBs?

- $\xi < 8$ from GRB 080916C (z=4.35)

  Abdo, A. A. et al., (Fermi LAT Collaboration), Science, **323**, 5922, 1688, (2009)

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BUT

$$\xi = \frac{E_{\mathrm{P}}}{E_{\mathrm{QG},1}} \gtrsim 1$$

Ellis, J. et al., Phys.Lett.B, **665**, 412, (2008)

Zloshchastiev, K. G., arXiv:0906.4282, (2009)

What about GRBs?

LETTERS

infrared photons[21,22], and fuzziness of radio or optical images of distant extragalactic sources[23–25]. Our stringent photon dispersion limit strongly disfavours models of Planck scale physics in which the quantum nature of space–time causes a linear variation of the speed of light with photon energy.

Received 12 August; accepted 12 October 2009.
Published online 28 October 2009.

1. Wheeler, J. A. & Ford, K. W. *Geons, Black Holes, and Quantum Foam: A Life in Physics* (W. W. Norton and Company, 1998).
2. Amelino-Camelia, G., Ellis, J., Mavromatos, N. E., Nanopoulos, D. V. & Sarkar, S. Tests of quantum gravity from observations of gamma-ray bursts. *Nature* **393**, 763–765 (1998).
3. Mattingly, D. Modern tests of Lorentz invariance. *Living Rev. Relativity* **8**, 5–84

Everything together

Source(s)	Experiment	Method	Results (95% CL limits)
GRB 021206	*RHESSI*	Fit + mean arrival time in a spike	$M_{QG}^{l} > 1.8 \times 10^{17}$ GeV
GRB 080916C	*Fermi* GBM + LAT	associating a 13 GeV photon with the trigger time	$M_{QG}^{l} > 1.3 \times 10^{18}$ GeV
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GRB 090510	*Fermi* GBM + LAT	associating a 31 GeV photon with the start of any observed emission, DisCan	$M_{QG}^{l} > 1.5 \times 10^{19}$ GeV
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9 GRBs	BATSE + OSSE	wavelets	$M_{QG}^{l} > 0.7 \times 10^{16}$ GeV
			$M_{QG}^{q} > 2.9 \times 10^{6}$ GeV
15 GRBs	*HETE-2*	wavelets	$M_{QG}^{l} > 0.4 \times 10^{16}$ GeV
17 GRBs	*INTEGRAL*	likelihood	$M_{QG}^{l} > 3.2 \times 10^{11}$ GeV
35 GRBs	BATSE + *HETE-2* + *Swift*	wavelets	$M_{QG}^{l} > 1.4 \times 10^{16}$ GeV
Mrk 421	Whipple	likelihood	$M_{QG}^{l} > 0.4 \times 10^{17}$ GeV
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Current LIV status

Collecting all the AGN and GRBs

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Collecting all the AGN and GRBs

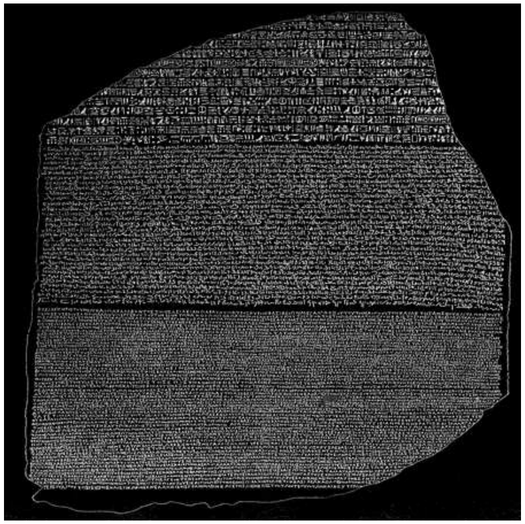
$$\Delta t \approx \xi \left(\frac{\Delta E}{E_{\mathrm{P}}} \right) \frac{L}{c_0} \Rightarrow \frac{\Delta t}{\Delta E} \approx \xi \frac{1}{E_{\mathrm{P}}} \frac{L}{c_0}$$

Current LIV status

Collecting all the AGN and GRBs

`Smoking Gun'

`Rosetta Stone'

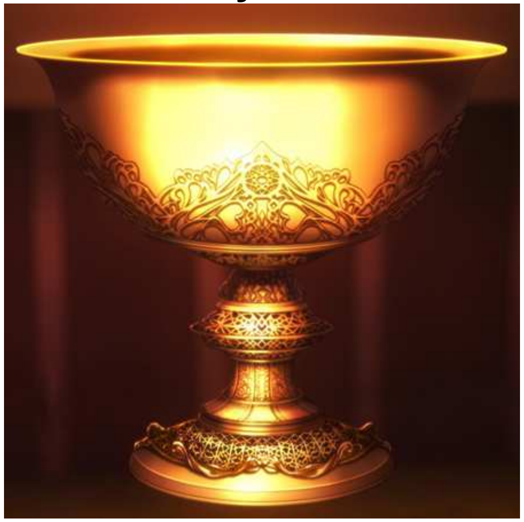


`Tip of the Iceberg'

Current LIV status

Collecting all the AGN and GRBs

`Holy Grail'



Current LIV status

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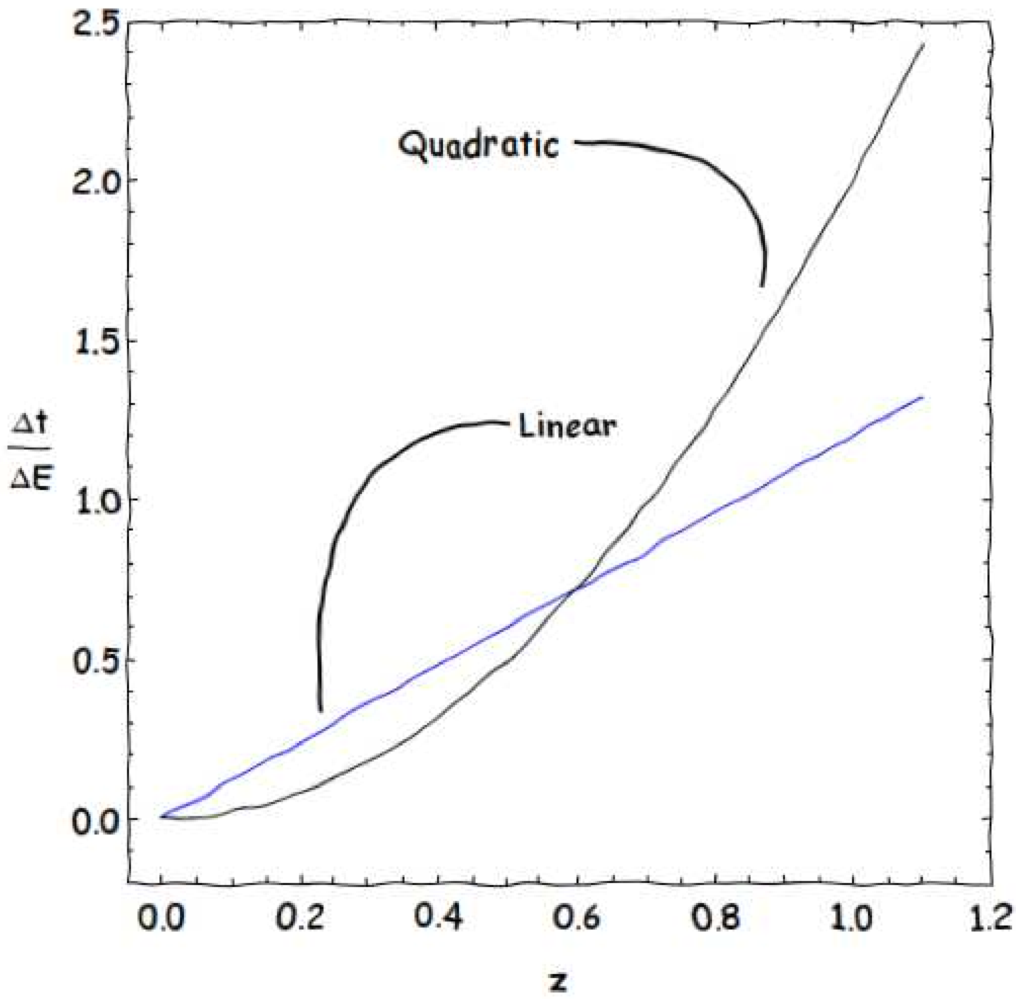

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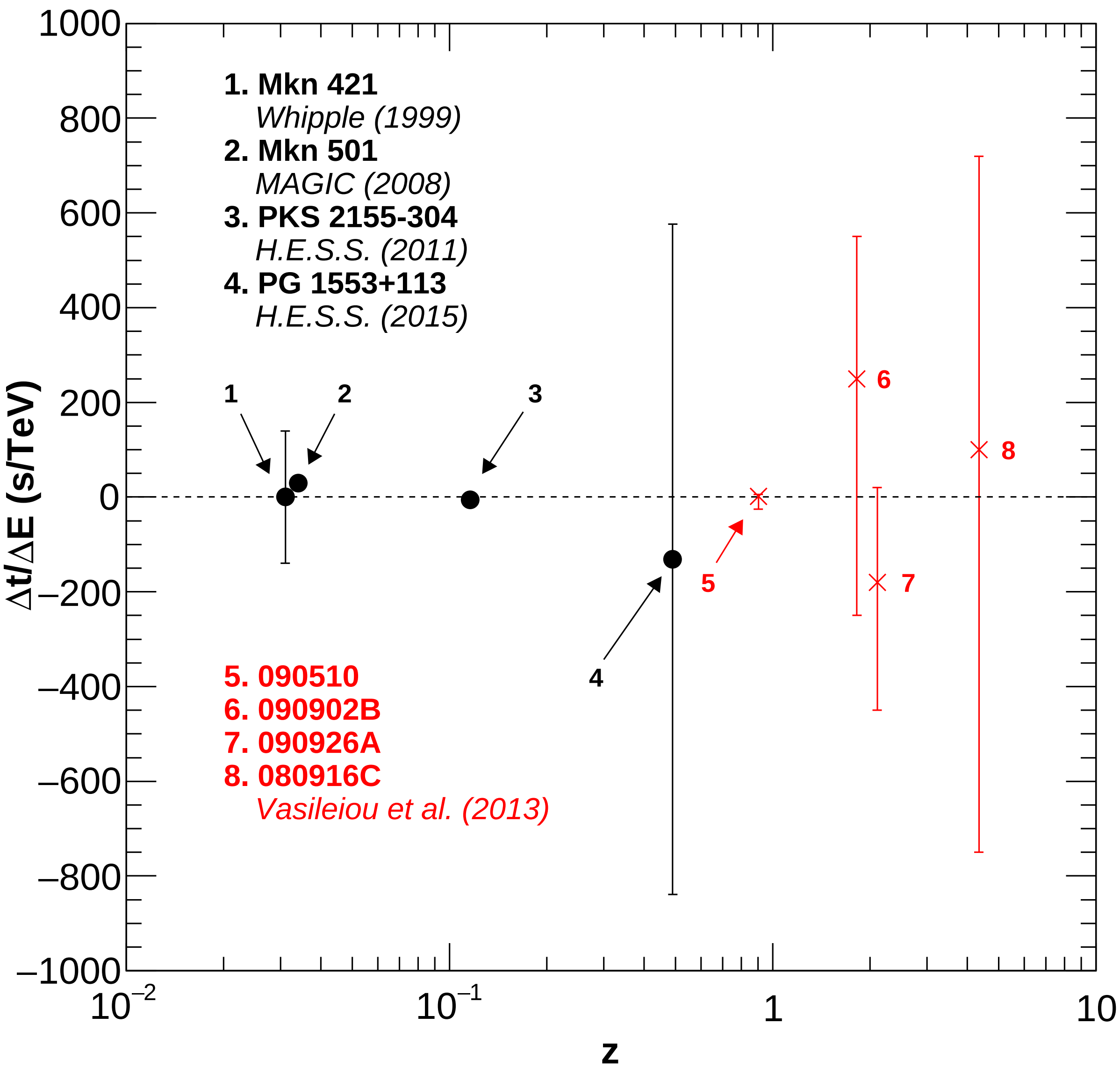
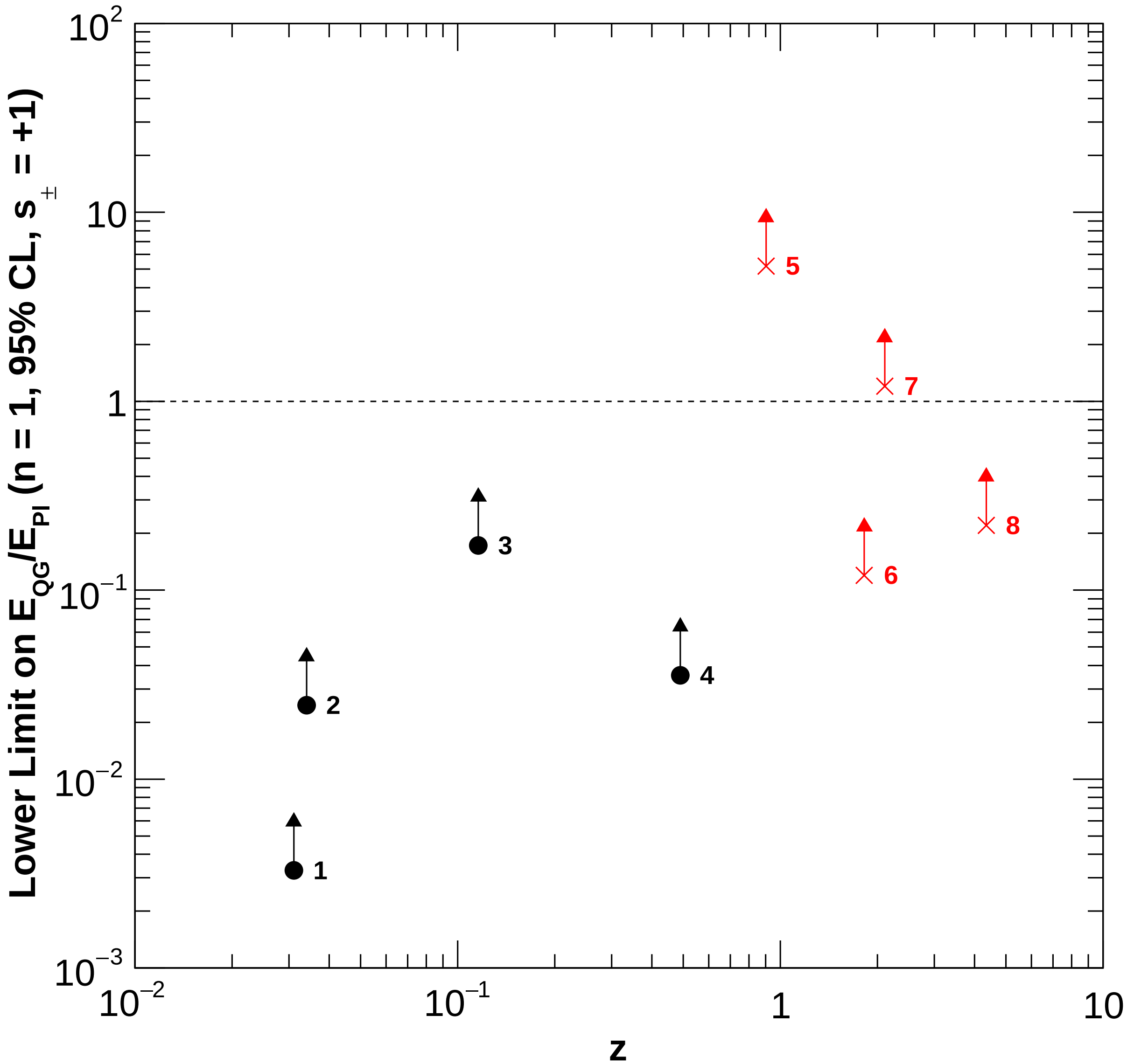


J. Bolmont et al., Nucl. Instr. Meth. Phys., **742**, 165, (2015)

Current LIV status



J. Bolmont et al., Nucl. Instr. Meth. Phys., **742**, 165, (2015)

Keep in mind...

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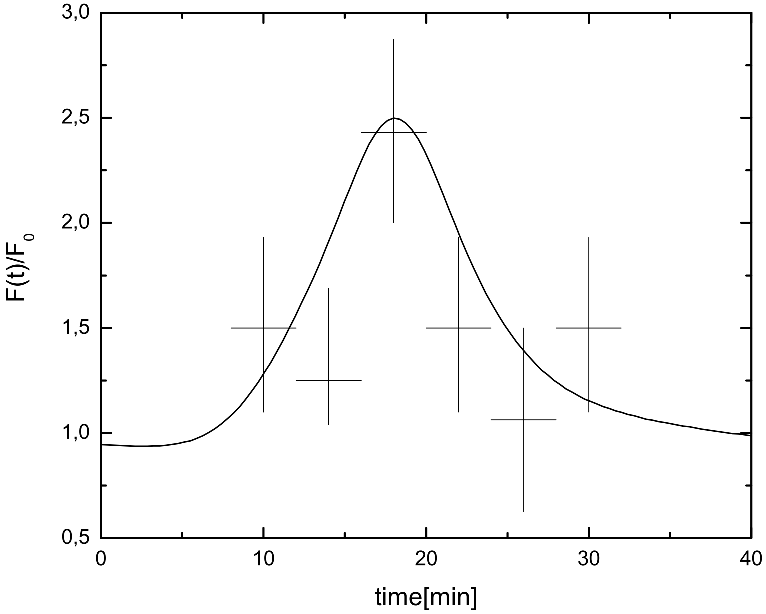
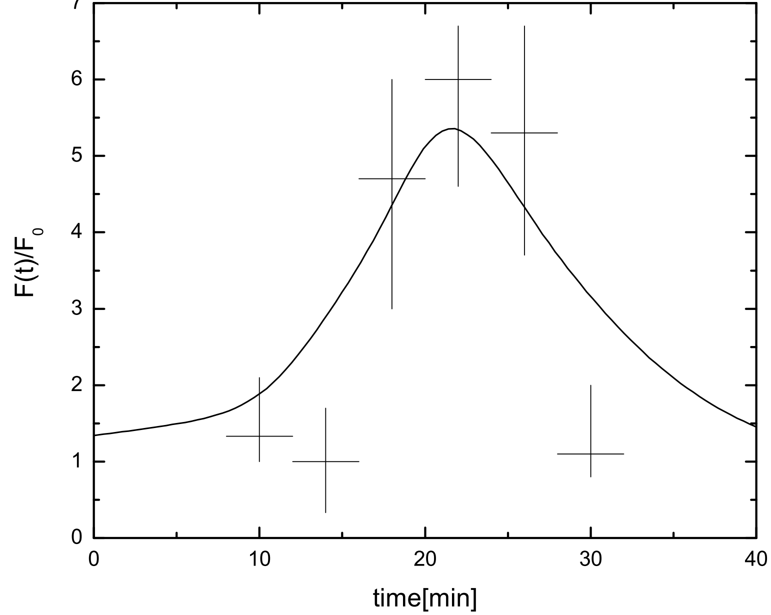
0.15–0.25 TeV



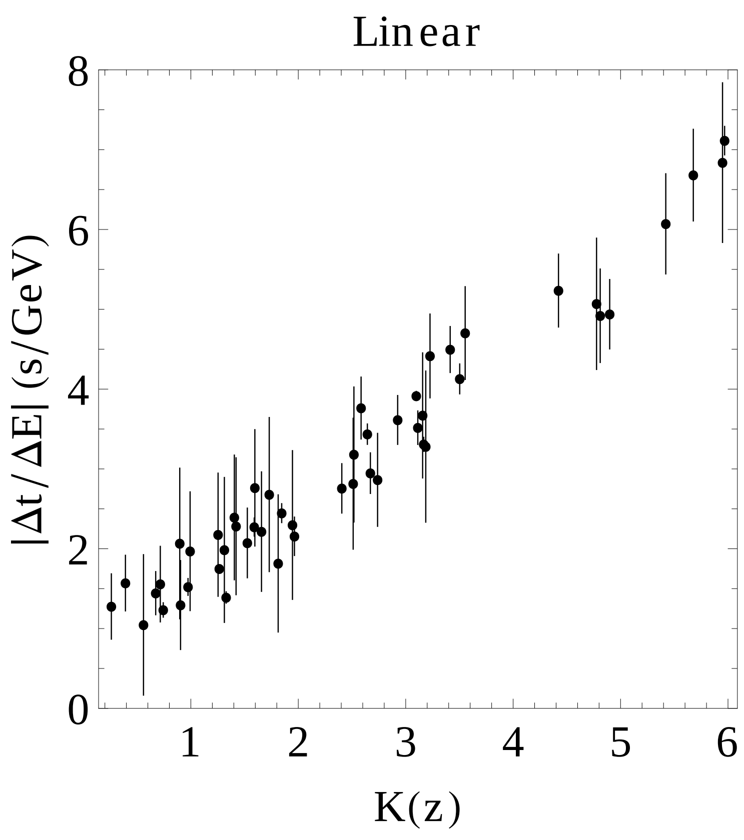
1.2–10 TeV

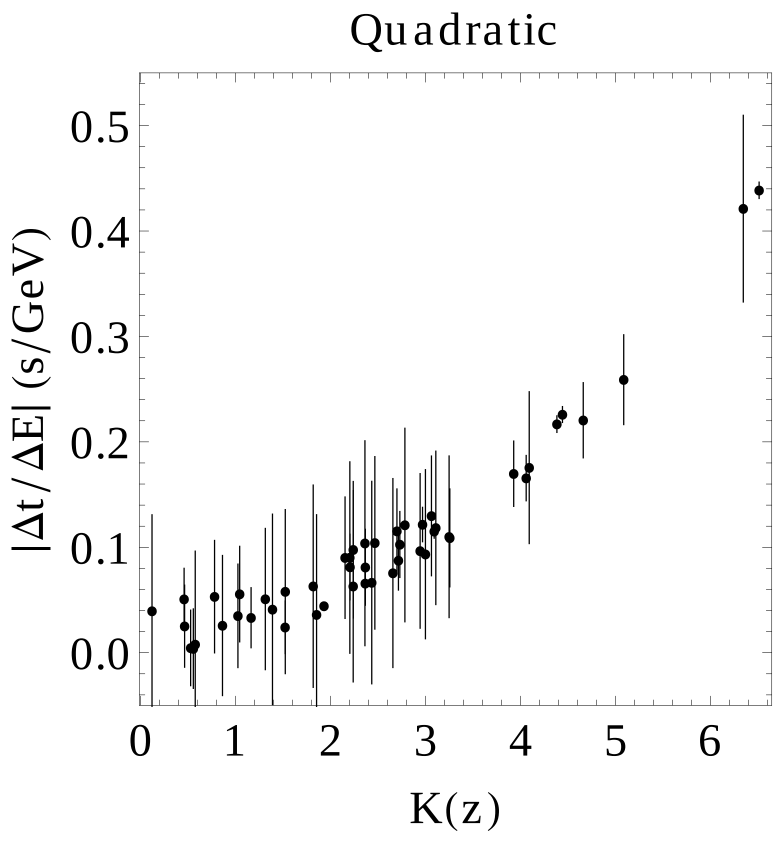


Mastichiadis, A. et al., A&A, **491**, 2, L37–L40, (2008)

Keep in mind...

- Energy dependent time-lags can be induced intrinsically in the source.
- Population studies are necessary: $\Delta t(z)$





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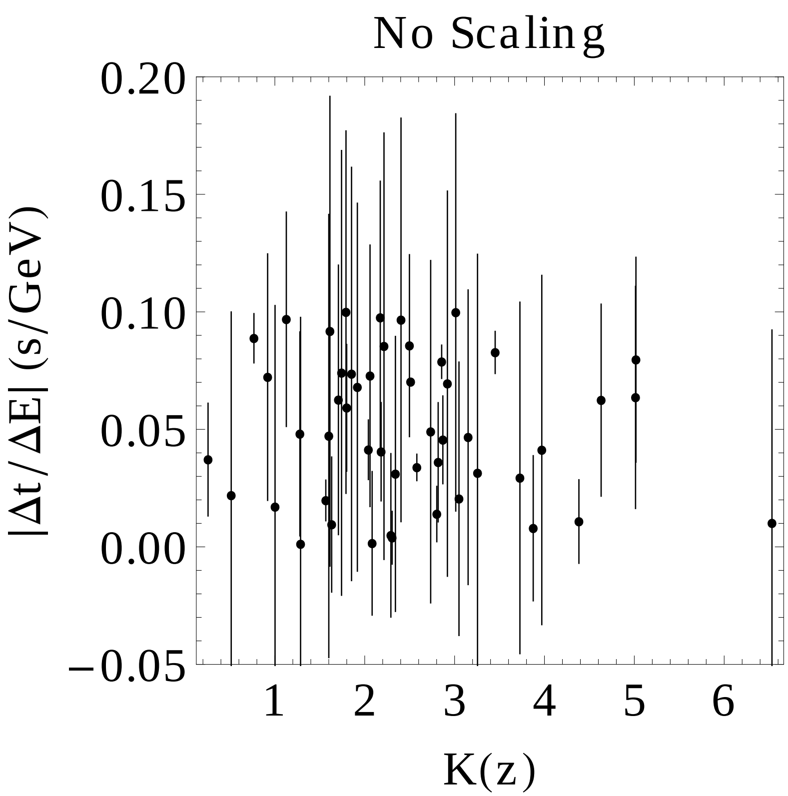

Keep in mind...

- Energy dependent time-lags can be induced intrinsically in the source.
- Population studies are necessary: $\Delta t(z)$
- Main emphasis on blazars: They are always there!!!

The future

Recap

- Several theories predict that vacuum may exhibit light-dispersional effects in Planck scales.

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Recap

- Several theories predict that vacuum may exhibit light-dispersional effects in Planck scales.
- Tiny variations in photon speed, accumulated over cosmological distances, can be unveiled from variable VHE BH emission.
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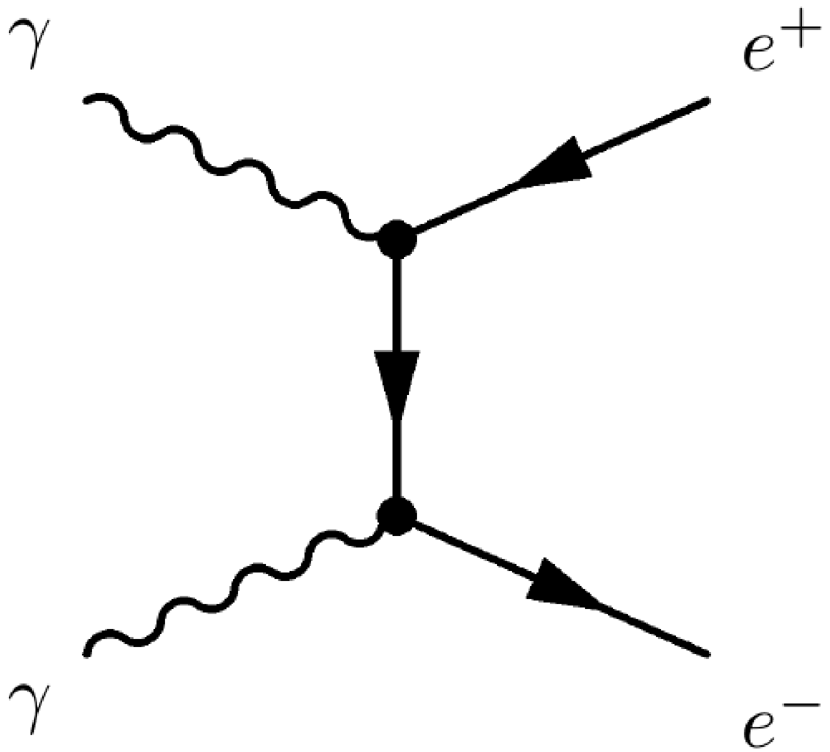
Recap

- Several theories predict that vacuum may exhibit light-dispersional effects in Planck scales.
- Tiny variations in photon speed, accumulated over cosmological distances, can be unveiled from variable VHE BH emission.
- Up to now, there is <u>no robust evidence</u> from an astrophysical point of view.
- Energy dependent time-lags can be produced in the source.
- Large source samples needed to reject or determine the existence of possible LIV.

Pair production

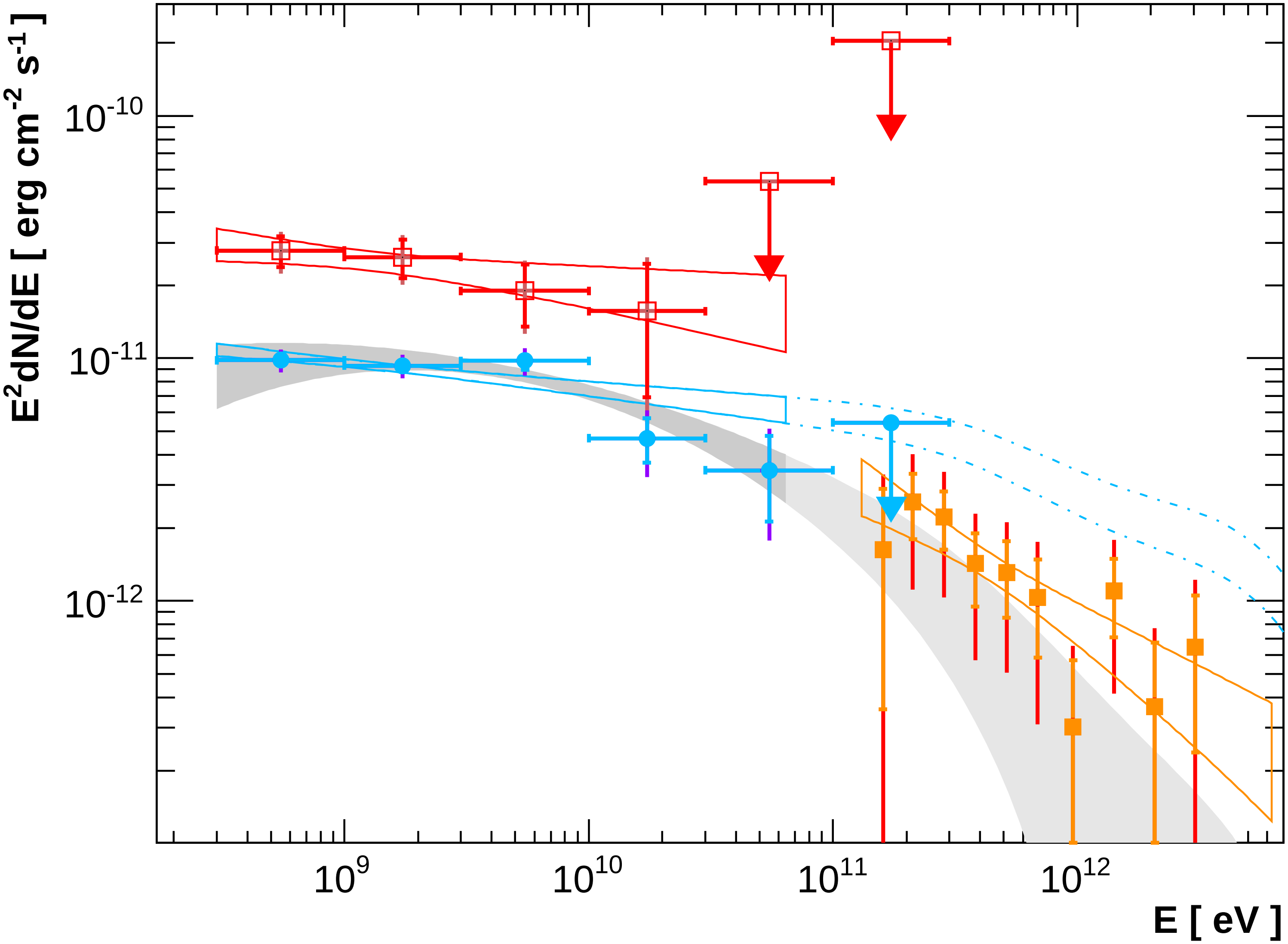
$$E^2 = p^2c^2 + m^2c^4$$



Pair production

$$E^2 = p^2c^2 + m^2c^4 + E^2\left(\frac{E}{E_{\mathrm{P}}}\right)^{\xi}$$

Pair production



H.E.S.S.& FERMI Collaborations, A&A, **573**, A31, (2015)

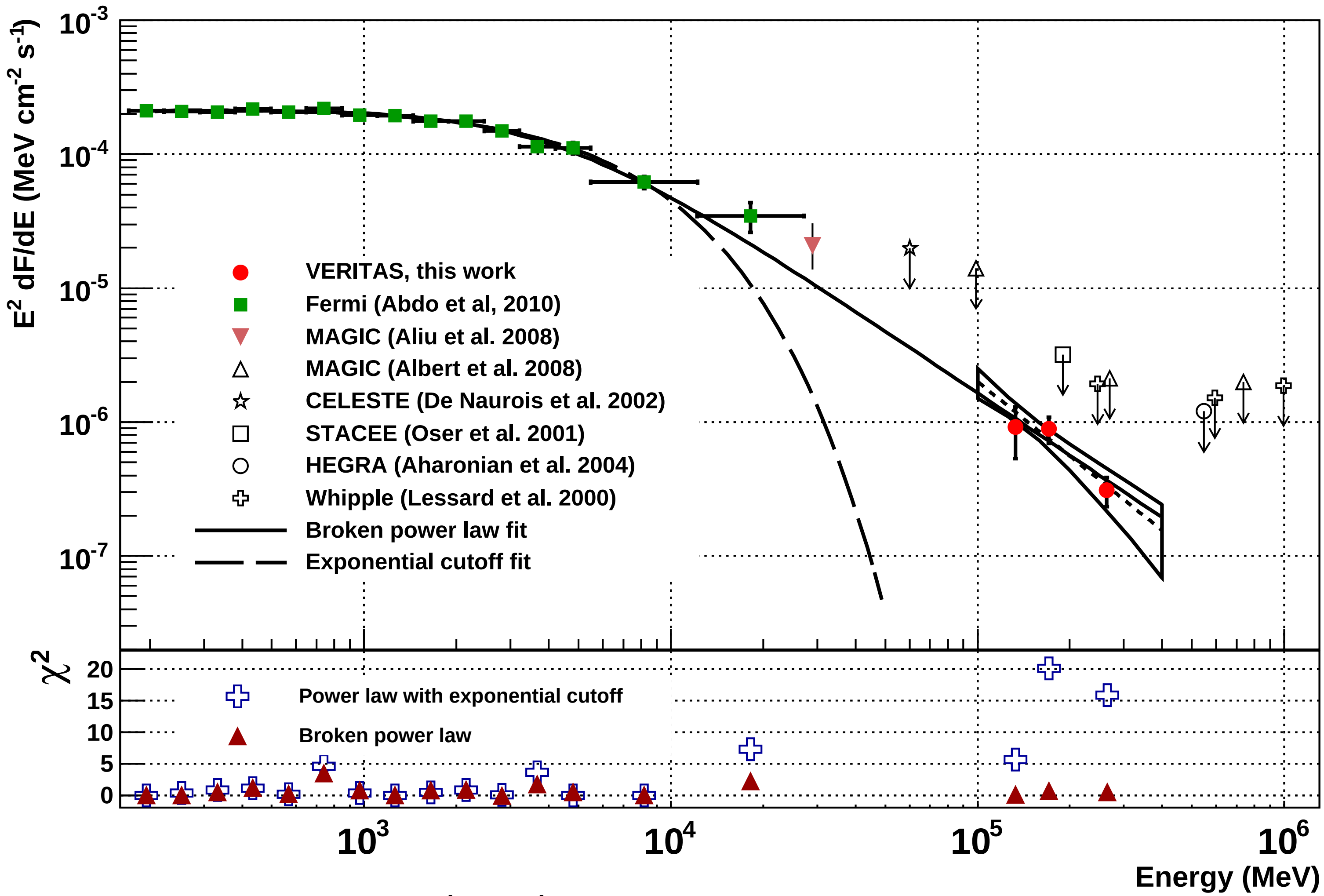
Quadratic term

$$\frac{\Delta t}{\Delta E^2} \approx \frac{3\zeta}{2\mathrm{E}_{\mathrm{P}}^2 \mathrm{H}_0} \int_0^z \frac{(1+z')^2\, dz'}{\sqrt{\Omega_m (1+z')^3 + \Omega_\Lambda}}$$

Quadratic term

$$\frac{\Delta t}{\Delta E^2} \approx \frac{3\zeta}{2\mathrm{E}_{\mathrm{P}}^2\mathrm{H}_0} L$$

Quadratic term



McCann, A. ICRC, **7**, 208, (2011)