Non-Singular Method of Fundamental Solutions and its Application to Two Dimensional Elasticity Problems

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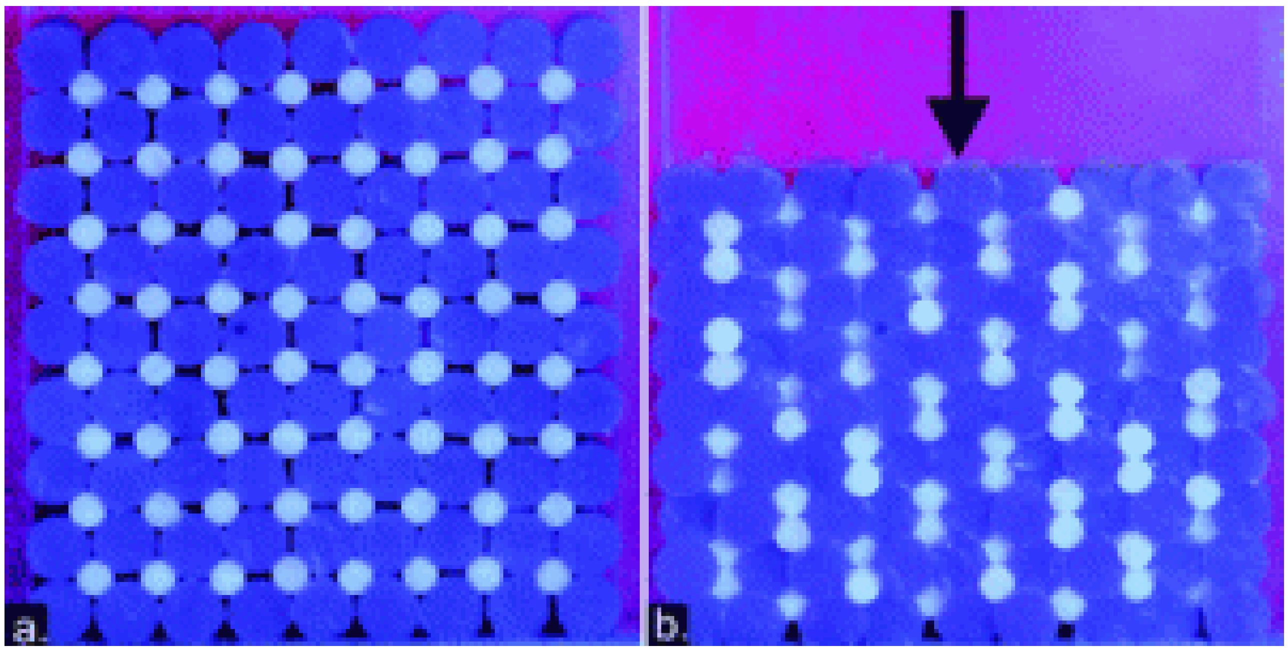
Scope

- **Motivation**
- **Introduction**
  - **Previous Applications of MFS**
- **Governing Equations of Elasticity**
- **Solution Procedure**
  - **MFS and NMFS**
- **Numerical Examples**
- **Conclusions**

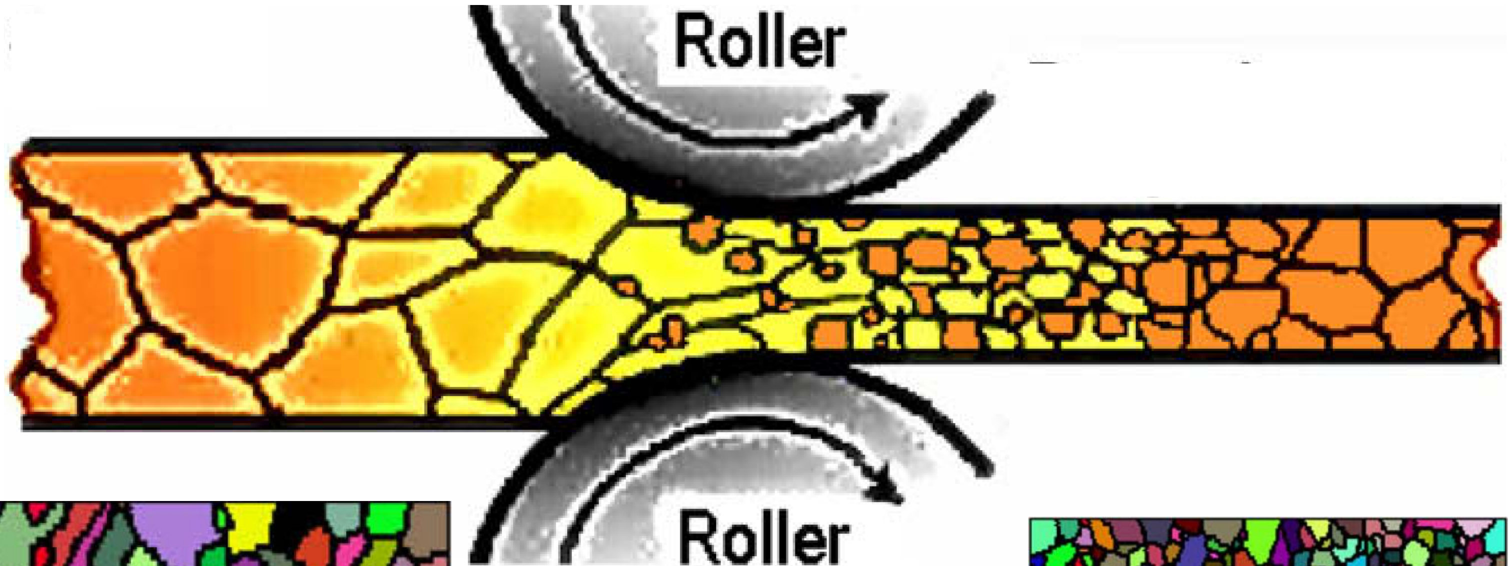
Motivation

What we want to know: displacement of object subject to external forces



Before deformation After deformation

Motivation



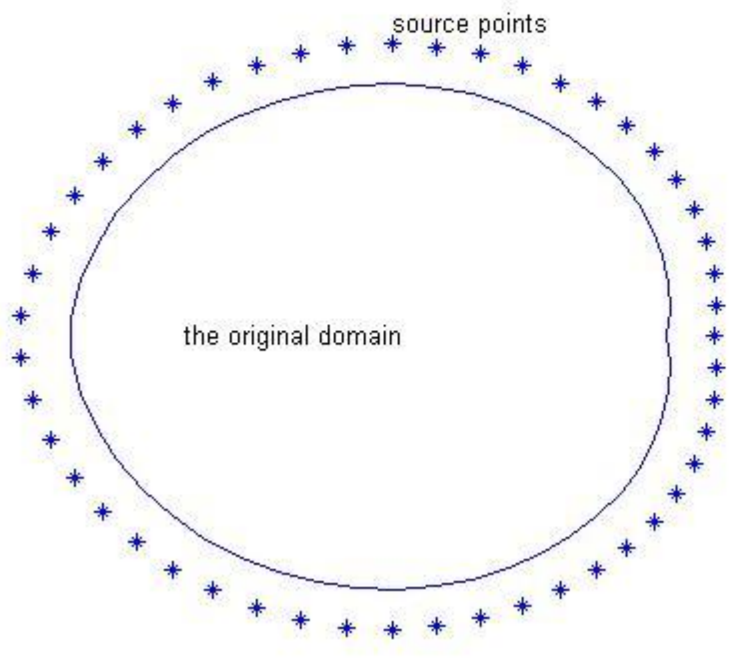
Initial microstructure

Final microstructure

The problem will be solved by MFS-type solution procedure.

Introduction

- Trefftz-type technique
- Approximating the solution of the problem by a linear combination of fundamental solutions
- Singularities, so source points have to be located outside the domain
- Boundary conditions
- Determining the unknown coefficients and the coordinates of the source points



Introduction

- Method widely used in linear elasticity.
  - Kupradze and Aleksidze, 1964. First used in solid mechanics
  - Smyrlis, 2006. Expand the result of Kupradze and Aleksidze
  - Maharejin, 1985. Anisotropic media
  - Berger and Karageorghis, 2001. Application to anisotropic problems
  - Redekop and Thompson, 1983. Axisymmetric problems
  - Marin and Lesnic, 2004. Cauchy problems

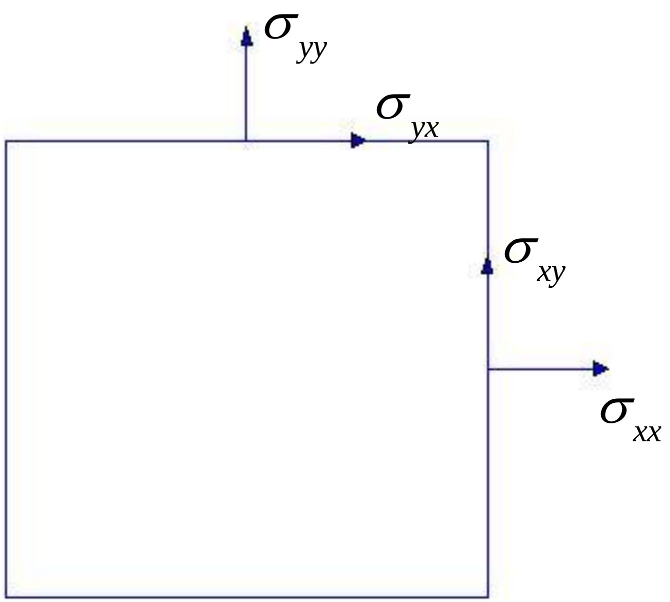
Introduction

- Traditional MFS
- Problems with the fictitious boundary with source points
- The source points can be placed on the real boundary directly
  - Young et al. 2005, 2007
  - Chen K. H. et al. 2006
  - Šarler, 2008, 2009
  - Chen and Wang, 2010
- A modified method: Liu, 2010
  - Used for potential problems
  - Integration of the fundamental solutions
  - Area-distributed sources cover the source points
  - Determining of the diagonal elements: Šarler, 2008, 2009

Governing Equations of Elasticity

- The state of stress at a point can be described by stress components formed on an element with sides δx and δy
- Symmetry:
  $\sigma_{xy} = \sigma_{yx}$
- The first suffix

indicates the indication of the normal to the plane on which the stress acts. The second suffix indicates the indication in which the stress acts.



Governing Equations of Elasticity

The equations of equilibrium for a two-dimensional system subjected to external forces and body forces b_x and b_y are given as

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}+b_x=0$$

$$\frac{\partial \sigma_{yx}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+b_y=0$$

where

$$b_x=ma_x \qquad b_y=ma_y$$

m is quality and a_x and a_y are accelerations along directions of x-axis and y-axis, rectively.

When there is no body force (b_x =0 and b_y=0):

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}=0$$

$$\frac{\partial \sigma_{yx}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}=0$$

Governing Equations of Elasticity

We introduce liner elasticity by Hooke's law .

$$\sigma_{xx} = \lambda(\varepsilon_{xx} + \varepsilon_{yy}) + 2\mu\varepsilon_{xx} \qquad \sigma_{xy} = 2\mu\varepsilon_{xy}$$

$$\sigma_{yy} = \lambda(\varepsilon_{xx} + \varepsilon_{yy}) + 2\mu\varepsilon_{yy} \qquad \sigma_{yx} = 2\mu\varepsilon_{yx}$$

where $\lambda = 2\mu\nu / (1-2\nu)$, ν is Poisson's ratio, $\mu = E/2(1-\nu)$ is the shear modulus of elasticity and E is a constant known as the modulus of elasticity or Young's modulus.

The definition of strain is

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad i, j = x, y$$

Governing Equations of Elasticity

Navier's equations without body force for isotropic elasticity are

$$\frac{2\mu\nu}{1-2\nu}\frac{\partial^2 u_x(\mathbf{p})}{\partial p_x^{\ 2}} + \mu\frac{\partial^2 u_x(\mathbf{p})}{\partial p_y^{\ 2}} + \frac{\mu}{1-2\nu}\frac{\partial^2 u_y(\mathbf{p})}{\partial p_x \partial p_y} = 0$$

$$\frac{2\mu\nu}{1-2\nu}\frac{\partial^2 u_y(\mathbf{p})}{\partial p_y^{\ 2}} + \mu\frac{\partial^2 u_y(\mathbf{p})}{\partial p_x^{\ 2}} + \frac{\mu}{1-2\nu}\frac{\partial^2 u_x(\mathbf{p})}{\partial p_x \partial p_y} = 0 \qquad \mathbf{p} \in \Omega$$

where $\mathbf{p} = p_x\mathbf{i}_x + p_y\mathbf{i}_y$ and p_x, p_y are 2D Cartesian coordinates with base vectors $\mathbf{i}_x, \mathbf{i}_y$

Governing Equations of Elasticity

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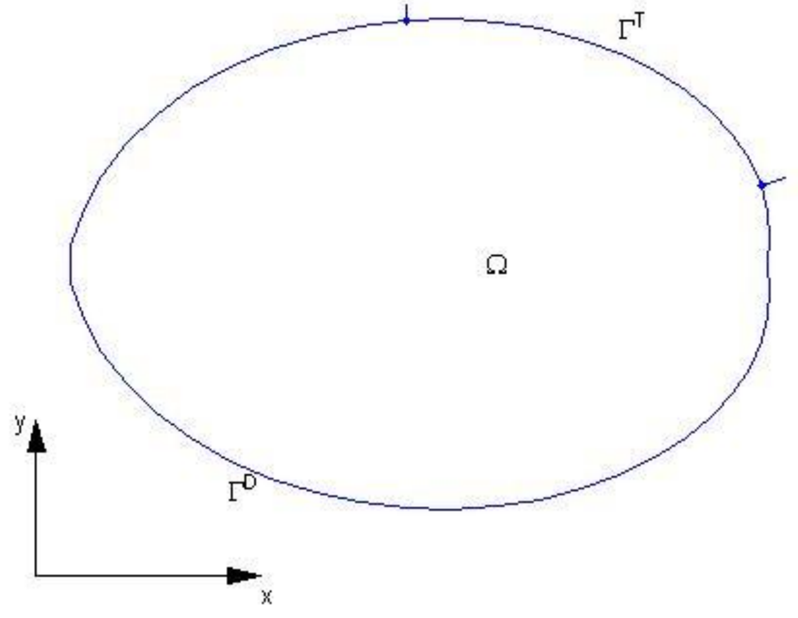
’ The boundary conditions are

$$u_i(\mathbf{p}) = \bar{u}_i(\mathbf{p}); \quad \mathbf{p} \in \Gamma^D \text{(Displacement B.C.)}$$

$$t_i(\mathbf{p}) = \bar{t}_i(\mathbf{p}); \quad \mathbf{p} \in \Gamma^T \text{(Traction B.C.)}$$

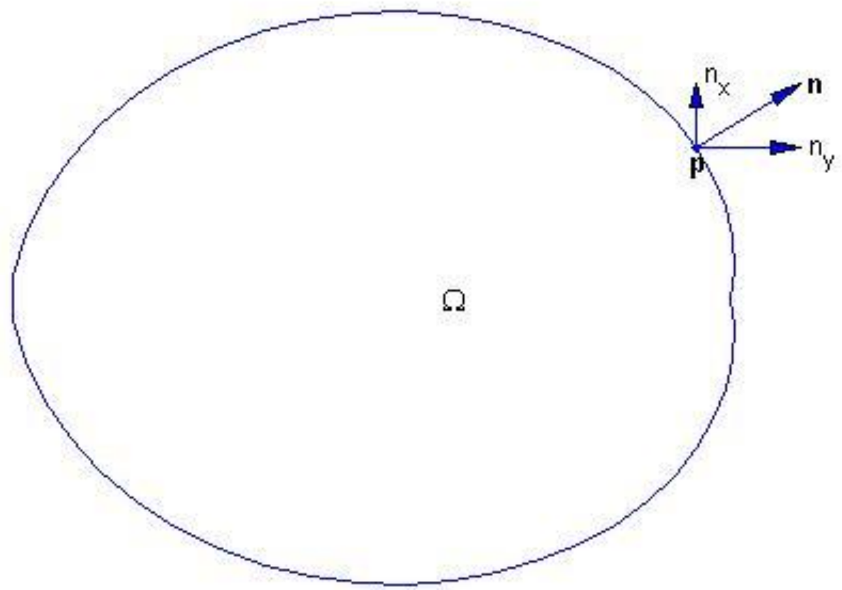
$$i = x, y$$



Governing Equations of Elasticity

The definition of tractions is

$$t_i = \sigma_{ii} n_i + \sigma_{ij} n_j \quad i, j = x, y \text{ and } j \neq i$$



Solution Procedure：MFS

The Kelvin's fundamental solution of elastostatics is

$$U_{ij}(\mathbf{p},\mathbf{s})=\frac{1}{8\pi\mu(1-\nu)}\left\{(3-4\nu)\log(\frac{1}{r})\delta_{ij}+\frac{(p_i-s_i)(p_j-s_j)}{r^2}\right\},\qquad i,j=x,y$$

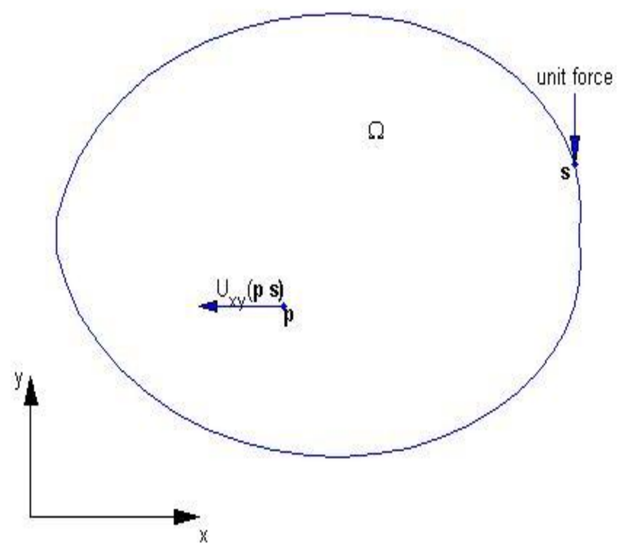
where U_{ij} (**p**, **s**) represents the displacement in the *i* direction at point **p** due to a unit point force acting in the *j* direction at **s**, $r=\sqrt{(p_x-q_x)^2+(p_y-q_y)^2}$ is the distance between the point **p** and **s.**

$$U_{xx}(\mathbf{p},\mathbf{s})=\frac{1}{8\pi\mu(1-\nu)}\left\{(3-4\nu)\log(\frac{1}{r})+\frac{(p_x-s_x)^2}{r^2}\right\}$$

$$U_{xy}(\mathbf{p},\mathbf{s})=U_{yx}(\mathbf{p},\mathbf{s})=\frac{1}{8\pi\mu(1-\nu)}\frac{(p_x-s_x)(p_y-s_y)}{r^2}$$

$$U_{yy}(\mathbf{p},\mathbf{s})=\frac{1}{8\pi\mu(1-\nu)}\left\{(3-4\nu)\log(\frac{1}{r})+\frac{(p_y-s_y)^2}{r^2}\right\}$$

$$\mathbf{p},\mathbf{s}\in\Omega$$



Solution Procedure：MFS

By using the definition of traction, we obtain

$$T_{xx}(\mathbf{p},\mathbf{s})=\left(\frac{2\mu(1-\nu)}{1-2\nu}\frac{\partial U_{xx}(\mathbf{p},\mathbf{s})}{\partial p_x}+\frac{2\mu\nu}{1-2\nu}\frac{\partial U_{yx}(\mathbf{p},\mathbf{s})}{\partial p_y}\right)n_x+\left(\mu\frac{\partial U_{xx}(\mathbf{p},\mathbf{s})}{\partial p_y}+\mu\frac{\partial U_{yx}(\mathbf{p},\mathbf{s})}{\partial p_x}\right)n_y$$

$$T_{xy}(\mathbf{p},\mathbf{s})=\left(\frac{2\mu(1-\nu)}{1-2\nu}\frac{\partial U_{xy}(\mathbf{p},\mathbf{s})}{\partial p_x}+\frac{2\mu\nu}{1-2\nu}\frac{\partial U_{yy}(\mathbf{p},\mathbf{s})}{\partial p_y}\right)n_x+\left(\mu\frac{\partial U_{xy}(\mathbf{p},\mathbf{s})}{\partial p_y}+\mu\frac{\partial U_{yy}(\mathbf{p},\mathbf{s})}{\partial p_x}\right)n_y$$

$$T_{yx}(\mathbf{p},\mathbf{s})=\left(\mu\frac{\partial U_{yx}(\mathbf{p},\mathbf{s})}{\partial p_x}+\mu\frac{\partial U_{xx}(\mathbf{p},\mathbf{s})}{\partial p_y}\right)n_x+\left(\frac{2\mu(1-\nu)}{1-2\nu}\frac{\partial U_{yx}(\mathbf{p},\mathbf{s})}{\partial p_y}+\frac{2\mu\nu}{1-2\nu}\frac{\partial U_{xx}(\mathbf{p},\mathbf{s})}{\partial p_x}\right)n_y$$

$$T_{yy}(\mathbf{p},\mathbf{s})=\left(\mu\frac{\partial U_{yy}(\mathbf{p},\mathbf{s})}{\partial p_x}+\mu\frac{\partial U_{xy}(\mathbf{p},\mathbf{s})}{\partial p_y}\right)n_x+\left(\frac{2\mu(1-\nu)}{1-2\nu}\frac{\partial U_{yy}(\mathbf{p},\mathbf{s})}{\partial p_y}+\frac{2\mu\nu}{1-2\nu}\frac{\partial U_{xy}(\mathbf{p},\mathbf{s})}{\partial p_x}\right)n_y$$

Solution Procedure：MFS

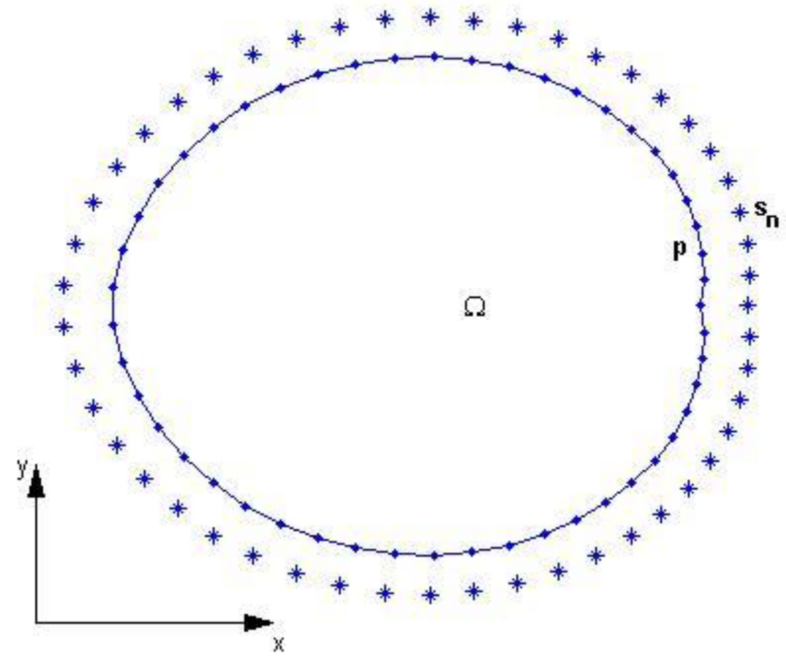
In MFS, the displacement field is represented by N fundamental solutions in source points $\mathbf{s}_n$, n=1,2, …,N. The displacements and the tractions are approximated by a linear combination of fundamental solutions as follows:

$$u_x(\mathbf{p}) = \sum_{n=1}^{N} U_{xx}(\mathbf{p},\mathbf{s}_n)\,\alpha_n + \sum_{n=1}^{N} U_{xy}(\mathbf{p},\mathbf{s}_n)\,\beta_n$$

$$u_y(\mathbf{p}) = \sum_{n=1}^{N} U_{yx}(\mathbf{p},\mathbf{s}_n)\,\alpha_n + \sum_{n=1}^{N} U_{yy}(\mathbf{p},\mathbf{s}_n)\,\beta_n$$

$$t_x(\mathbf{p}) = \sum_{n=1}^{N} T_{xx}(\mathbf{p},\mathbf{s}_n)\,\alpha_n + \sum_{n=1}^{N} T_{xy}(\mathbf{p},\mathbf{s}_n)\,\beta_n$$

$$t_y(\mathbf{p}) = \sum_{n=1}^{N} T_{yx}(\mathbf{p},\mathbf{s}_n)\,\alpha_n + \sum_{n=1}^{N} T_{yy}(\mathbf{p},\mathbf{s}_n)\,\beta_n$$



Solution Procedure：MFS

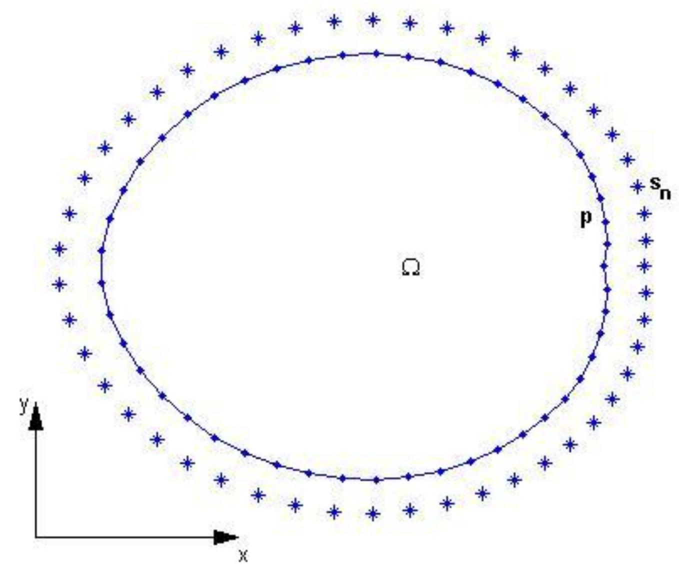
Let us place N points $\mathbf{p}_{n,}$ n=1,2, …,N. on the boundary

$$\bar{u}_x(\mathbf{p}) = \sum_{n=1}^{N} U_{xx}(\mathbf{p},\mathbf{s}_n)\,\alpha_n + \sum_{n=1}^{N} U_{xy}(\mathbf{p},\mathbf{s}_n)\,\beta_n$$

$$\bar{u}_y(\mathbf{p}) = \sum_{n=1}^{N} U_{yx}(\mathbf{p},\mathbf{s}_n)\,\alpha_n + \sum_{n=1}^{N} U_{yy}(\mathbf{p},\mathbf{s}_n)\,\beta_n$$

$$\bar{t}_x(\mathbf{p}) = \sum_{n=1}^{N} T_{xx}(\mathbf{p},\mathbf{s}_n)\,\alpha_n + \sum_{n=1}^{N} T_{xy}(\mathbf{p},\mathbf{s}_n)\,\beta_n$$

$$\bar{t}_y(\mathbf{p}) = \sum_{n=1}^{N} T_{yx}(\mathbf{p},\mathbf{s}_n)\,\alpha_n + \sum_{n=1}^{N} T_{yy}(\mathbf{p},\mathbf{s}_n)\,\beta_n$$



We introduce boundary condition indicators

$$\chi^D(\mathbf{p}) = \begin{cases} 1, & \mathbf{p} \in \Gamma^D \\ 0, & \mathbf{p} \in \Gamma^T \end{cases} \qquad \chi^T(\mathbf{p}) = \begin{cases} 1, & \mathbf{p} \in \Gamma^T \\ 0, & \mathbf{p} \in \Gamma^D \end{cases}$$

Solution Procedure： MFS

The coefficients are calculated from a system of algebraic equations

$$\begin{bmatrix} \mathbf{\Phi} & \mathbf{\Psi} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix} = \mathbf{b}$$

$$\sum_{k=1}^{N} (\phi_{kn}\alpha_n + \psi_{kn}\beta_n) = b_k \qquad k = 1, 2, \cdots N$$

$$\phi_{kn} = \chi^D(\mathbf{p}_k)U_{ix}(\mathbf{p}_k, \mathbf{s}_n) + \chi^T(\mathbf{p}_k)T_{ix}(\mathbf{p}_k, \mathbf{s}_n)$$

$$\psi_{kn} = \chi^D(\mathbf{p}_k)U_{iy}(\mathbf{p}_k, \mathbf{s}_n) + \chi^T(\mathbf{p}_k)T_{iy}(\mathbf{p}_k, \mathbf{s}_n)$$

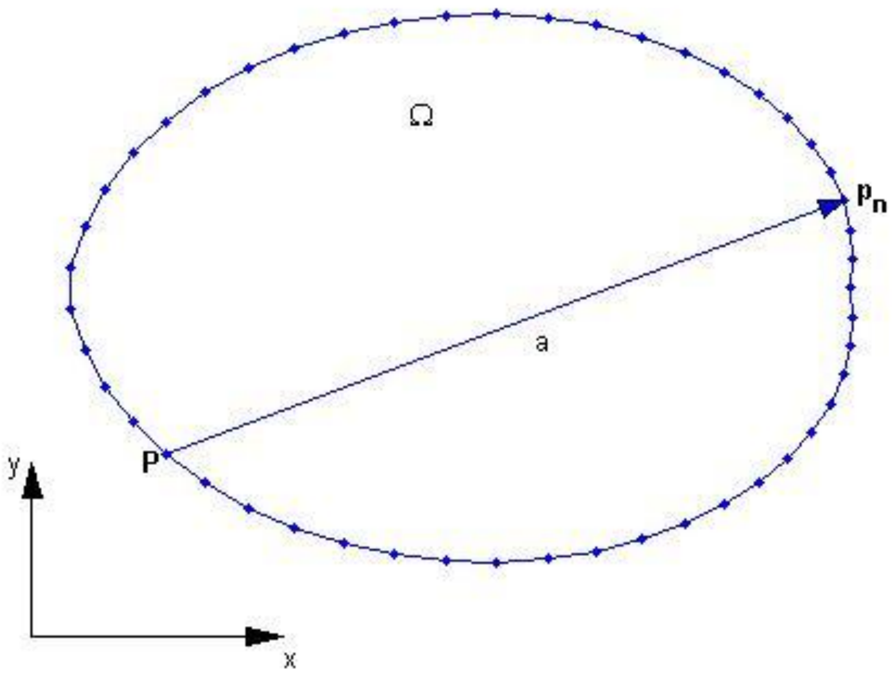
$$b_k = \chi^D(\mathbf{p}_k)\overline{u}_i(\mathbf{p}_k) + \chi^T(\mathbf{p}_k)\overline{t}_i(\mathbf{p}_k)$$

$$\begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi} & \mathbf{\Psi} \end{bmatrix}^{-1} \mathbf{b}$$

$$\begin{bmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{bmatrix} \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix} = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

Solution Procedure: NMFS

In the non-singular method of fundamental solutions, we place N distributed sources $\mathbf{s}_n$ at points $\mathbf{p}_{n,}$ n=1,2, …,N, on the boundary .



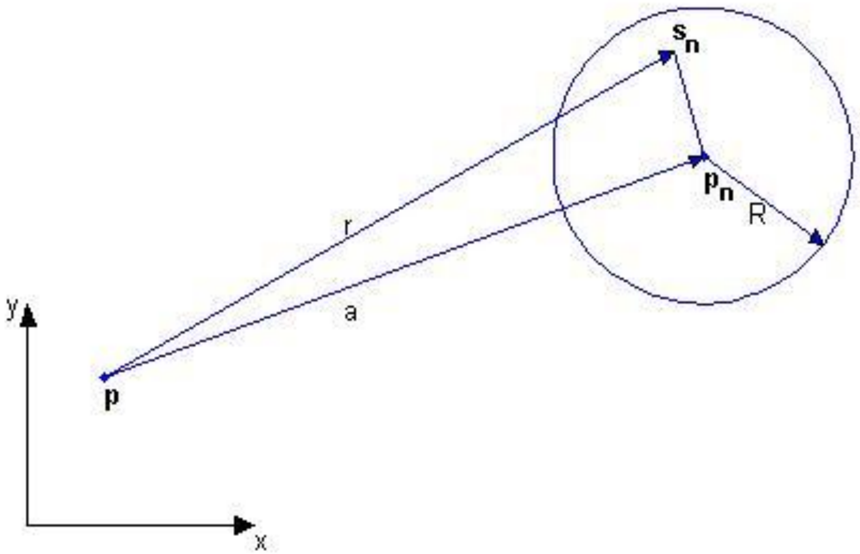
Solution Procedure: NMFS

The following two formulas satisfy the governing equation

$$u_x(\mathbf{p}) = \sum_{n=1}^{N} \int_{A(\mathbf{p}_n)} U_{xx}(\mathbf{p}, \mathbf{s}_n) dA(\mathbf{s}_n)\, \alpha_n + \sum_{n=1}^{N} \int_{A(\mathbf{p}_n)} U_{xy}(\mathbf{p}, \mathbf{s}_n) dA(\mathbf{s}_n)\, \beta_n$$

$$u_y(\mathbf{p}) = \sum_{n=1}^{N} \int_{A(\mathbf{p}_n)} U_{yx}(\mathbf{p}, \mathbf{s}_n) dA(\mathbf{s}_n)\, \alpha_n + \sum_{n=1}^{N} \int_{A(\mathbf{p}_n)} U_{yy}(\mathbf{p}, \mathbf{s}_n) dA(\mathbf{s}_n)\, \beta_n$$

We are using integration over point sources for smoothing out the singularity.



Solution Procedure: NMFS

When $\mathbf{p}_n$ is outside of $A(\mathbf{p}_n)$, we consider $r \approx a$ since $R << r$. So

$$\int_{A(P_n)} U_{ij}(\mathbf{p},\mathbf{s}_n)\,dA(\mathbf{s}_n) = \frac{R^2}{8\mu(1-\nu)}\left\{(3-4\nu)\log\left(\frac{1}{a}\right)\delta_{ij} + \frac{(p_i - p_{ni})(p_j - p_{nj})}{a^2}\right\} \qquad i, j = x, y$$

When $\mathbf{p}$ is inside of $A(\mathbf{p}_n)$,

$$\int_{A(\mathbf{p}_n)} \log\left(\frac{1}{r}\right) dA(\mathbf{s}_n) = -\int_0^{2\pi}\int_0^R \log(r)\cdot r\,dr\,d\theta = -2\pi\,\frac{1}{2}\left(\log(r)\cdot r\Big|_0^R - \int_0^R r\,dr\right) = \pi\log\left(\frac{1}{R}\right) + \frac{\pi R^2}{2}$$

$$\int_{A(\mathbf{p})} \frac{(p_x - s_{nx})^2}{r^2}\,dA(\mathbf{s}_n) = \int_0^{2\pi}\int_0^R \frac{r^2\cos^2(\theta)}{r^2}\cdot r\,dr\,d\theta = \frac{R^2}{2}\left(\frac{\sin(\theta)\cos(\theta)+\theta}{2}\right)\Bigg|_0^{2\pi} = \frac{\pi R^2}{2}$$

$$\int_{A(\mathbf{p})} \frac{(p_x - s_{nx})(p_y - s_{ny})}{r^2}\,dA(\mathbf{s}_n) = \int_0^{2\pi}\int_0^R \frac{r^2\cos(\theta)\sin(\theta)}{r^2}\cdot r\,dr\,d\theta = \frac{R^2}{2}\left(\frac{\cos^2(\theta)}{2}\right)\Bigg|_0^{2\pi} = 0$$

$$\int_{A(\mathbf{p})} \frac{(p_y - s_{ny})^2}{r^2}\,dA(\mathbf{s}_n) = \int_0^{2\pi}\int_0^R \frac{r^2\sin^2(\theta)}{r^2}\cdot r\,dr\,d\theta = \frac{R^2}{2}\left(\frac{-\sin(\theta)\cos(\theta)+\theta}{2}\right)\Bigg|_0^{2\pi} = \frac{\pi R^2}{2}$$

Solution Procedure: NMFS

$$\tilde{U}_{xx}(\mathbf{p},\mathbf{p}_n)=\int_{A(\mathbf{p}_n)}U_{xx}(\mathbf{p},\mathbf{s}_n)dA(\mathbf{s}_n)$$

$$=\begin{cases}\dfrac{R^2}{8\mu(1-\nu)}\left\{(3-4\nu)\log(\dfrac{1}{a})+\dfrac{(p_x-p_{nx})^2}{a^2}\right\}, & a>R\\ \dfrac{R^2(3-4\nu)}{8\mu(1-\nu)}\log(\dfrac{1}{R})+\dfrac{(3-4\nu)\left(R^2-a^2\right)}{16\mu(1-\nu)}+\dfrac{R^2-a^2}{16\mu(1-\nu)}+\dfrac{(p_x-p_{nx})^2}{8\mu(1-\nu)}, & a\le R\end{cases}$$

$$\tilde{U}_{xy}(\mathbf{p},\mathbf{p}_n)=\tilde{U}_{yx}(\mathbf{p},\mathbf{p}_n)=\int_{A(\mathbf{p}_n)}U_{xy}(\mathbf{p},\mathbf{s}_n)dA(\mathbf{s}_n)$$

$$=\begin{cases}\dfrac{R^2}{8\mu(1-\nu)}\dfrac{(p_x-p_{nx})(p_y-p_{ny})}{a^2}, & a>R\\ 0, & a\le R\end{cases}$$

$$\tilde{U}_{yy}(\mathbf{p},\mathbf{p}_n)=\int_{A(\mathbf{p}_n)}U_{yy}(\mathbf{p},\mathbf{s}_n)dA(\mathbf{s}_n)$$

$$=\begin{cases}\dfrac{R^2}{8\mu(1-\nu)}\left\{(3-4\nu)\log(\dfrac{1}{a})+\dfrac{(p_y-p_{ny})^2}{a^2}\right\}, & a>R\\ \dfrac{R^2(3-4\nu)}{8\mu(1-\nu)}\log(\dfrac{1}{R})+\dfrac{(3-4\nu)\left(R^2-a^2\right)}{16\mu(1-\nu)}+\dfrac{R^2-a^2}{16\mu(1-\nu)}+\dfrac{(p_y-p_{ny})^2}{8\mu(1-\nu)}, & a\le R\end{cases}$$

Solution Procedure: NMFS

The displacements are approximated by a linear combination of $\tilde{U}_{ij}\,(\mathbf{p}, \mathbf{p}_n)$ as follows:

$$u_x(\mathbf{p}) = \sum_{n=1}^{N} \tilde{U}_{xx}(\mathbf{p}, \mathbf{p}_n)\,\alpha_n + \sum_{n=1}^{N} \tilde{U}_{xy}(\mathbf{p}, \mathbf{p}_n)\,\beta_n$$

$$u_y(\mathbf{p}) = \sum_{n=1}^{N} \tilde{U}_{yx}(\mathbf{p}, \mathbf{p}_n)\,\alpha_n + \sum_{n=1}^{N} \tilde{U}_{yy}(\mathbf{p}, \mathbf{p}_n)\,\beta_n$$

Solution Procedure: NMFS

$$t_x(\mathbf{p}) = \sum_{n=1}^{N} \alpha_n \left\{ \left[\frac{2\mu(1-\nu)}{1-2\nu} \frac{\partial \tilde{U}_{xx}(\mathbf{p},\mathbf{p}_n)}{\partial p_x} + \frac{2\mu\nu}{1-2\nu} \frac{\partial \tilde{U}_{yx}(\mathbf{p},\mathbf{p}_n)}{\partial p_y} \right] n_x + \left[\mu \frac{\partial \tilde{U}_{xx}(\mathbf{p},\mathbf{p}_n)}{\partial p_y} + \mu \frac{\partial \tilde{U}_{yx}(\mathbf{p},\mathbf{p}_n)}{\partial p_x} \right] n_y \right\}$$
$$+ \sum_{n=1}^{N} \beta_n \left\{ \left[\frac{2\mu(1-\nu)}{1-2\nu} \frac{\partial \tilde{U}_{xy}(\mathbf{p},\mathbf{p}_n)}{\partial p_x} + \frac{2\mu\nu}{1-2\nu} \frac{\partial \tilde{U}_{yy}(\mathbf{p},\mathbf{p}_n)}{\partial p_y} \right] n_x + \left[\mu \frac{\partial \tilde{U}_{xy}(\mathbf{p},\mathbf{p}_n)}{\partial p_y} + \mu \frac{\partial \tilde{U}_{yy}(\mathbf{p},\mathbf{p}_n)}{\partial p_x} \right] n_y \right\}$$

$$t_y(\mathbf{p}) = \sum_{n=1}^{N} \alpha_n \left\{ \left[\mu \frac{\partial \tilde{U}_{yx}(\mathbf{p},\mathbf{p}_n)}{\partial p_x} + \mu \frac{\partial \tilde{U}_{xx}(\mathbf{p},\mathbf{p}_n)}{\partial p_y} \right] n_x + \left[\frac{2\mu(1-\nu)}{1-2\nu} \frac{\partial \tilde{U}_{yx}(\mathbf{p},\mathbf{p}_n)}{\partial p_y} + \frac{2\mu\nu}{1-2\nu} \frac{\partial \tilde{U}_{xx}(\mathbf{p},\mathbf{p}_n)}{\partial p_x} \right] n_y \right\}$$
$$+ \sum_{n=1}^{N} \beta_n \left\{ \left[\mu \frac{\partial \tilde{U}_{yy}(\mathbf{p},\mathbf{p}_n)}{\partial p_x} + \mu \frac{\partial \tilde{U}_{xy}(\mathbf{p},\mathbf{p}_n)}{\partial p_y} \right] n_x + \left[\frac{2\mu(1-\nu)}{1-2\nu} \frac{\partial \tilde{U}_{yy}(\mathbf{p},\mathbf{p}_n)}{\partial p_y} + \frac{2\mu\nu}{1-2\nu} \frac{\partial \tilde{U}_{xy}(\mathbf{p},\mathbf{p}_n)}{\partial p_x} \right] n_y \right\}$$

Solution Procedure: NMFS

The coefficients are calculated from a system of *N* algebraic equations

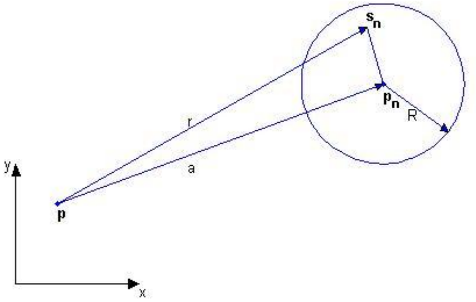
$$\begin{bmatrix}\boldsymbol{\Phi} & \boldsymbol{\Psi}\end{bmatrix}\begin{bmatrix}\boldsymbol{\alpha} \\ \boldsymbol{\beta}\end{bmatrix} = \mathbf{b}$$

$$\sum_{k=1}^{N}(\phi_{kn}\alpha_n + \psi_{kn}\beta_n) = b_k \qquad k = 1, 2, \cdots N$$

$$\phi_{kn} = \chi^D(\mathbf{p}_k)\tilde{U}_{ix}(\mathbf{p}_k, \mathbf{p}_n) + \chi^T(\mathbf{p}_k)\tilde{T}_{ix}(\mathbf{p}_k, \mathbf{p}_n)$$

$$\psi_{kn} = \chi^D(\mathbf{p}_k)\tilde{U}_{iy}(\mathbf{p}_k, \mathbf{p}_n) + \chi^T(\mathbf{p}_k)\tilde{T}_{iy}(\mathbf{p}_k, \mathbf{p}_n)$$

$$b_k = \chi^D(\mathbf{p}_k)\overline{u}_i(\mathbf{p}_k) + \chi^T(\mathbf{p}_k)\overline{t}_i(\mathbf{p}_k)$$



Solution Procedure: NMFS

The method proposed by Šarler 2009 is applied to determine the diagonal coefficients of matrix [Φ, Ψ]. We first assume a constant solution, e.g $\bar{u}_i(\mathbf{p}) = c$ everywhere. Then we can solve for the corresponding α_n^c and β_n^c, and use

$$\frac{\partial \bar{u}_i(\mathbf{p})}{\partial p_i} = \frac{\partial \bar{u}_i(\mathbf{p})}{\partial p_j} = 0, \qquad i, j = x, y$$

then

$$\frac{\partial \bar{u}_x(\mathbf{p})}{\partial p_x} = \sum_{n=1}^{N} \frac{\partial \tilde{U}_{xx}(\mathbf{p},\mathbf{p}_n)}{\partial p_x} \alpha_n^c + \sum_{n=1}^{N} \frac{\partial \tilde{U}_{xy}(\mathbf{p},\mathbf{p}_n)}{\partial p_x} \beta_n^c = 0$$

$$\frac{\partial \bar{u}_x(\mathbf{p})}{\partial p_y} = \sum_{n=1}^{N} \frac{\partial \tilde{U}_{xx}(\mathbf{p},\mathbf{p}_n)}{\partial p_y} \alpha_n^c + \sum_{n=1}^{N} \frac{\partial \tilde{U}_{xy}(\mathbf{p},\mathbf{p}_n)}{\partial p_y} \beta_n^c = 0$$

$$\frac{\partial \bar{u}_y(\mathbf{p})}{\partial p_x} = \sum_{n=1}^{N} \frac{\partial \tilde{U}_{yx}(\mathbf{p},\mathbf{p}_n)}{\partial p_x} \alpha_n^c + \sum_{n=1}^{N} \frac{\partial \tilde{U}_{yy}(\mathbf{p},\mathbf{p}_n)}{\partial p_x} \beta_n^c = 0$$

$$\frac{\partial \bar{u}_y(\mathbf{p})}{\partial p_y} = \sum_{n=1}^{N} \frac{\partial \tilde{U}_{yx}(\mathbf{p},\mathbf{p}_n)}{\partial p_y} \alpha_n^c + \sum_{n=1}^{N} \frac{\partial \tilde{U}_{yy}(\mathbf{p},\mathbf{p}_n)}{\partial p_y} \beta_n^c = 0$$

Solution Procedure: NMFS

We assume

$$\sum_{n=1}^{N} \frac{\partial \tilde{U}_{ij}(\mathbf{p},\mathbf{p}_n)}{\partial p_i} \alpha_n^c = 0$$

$$\sum_{n=1}^{N} \frac{\partial \tilde{U}_{ij}(\mathbf{p},\mathbf{p}_n)}{\partial p_i} \beta_n^c = 0 \qquad i, j = x, y$$

So when $\mathbf{p} = \mathbf{p}_m, m = 1, 2, \cdots, N$

$$\frac{\partial \tilde{U}_{jx}(\mathbf{p},\mathbf{p}_m)}{\partial p_i} = -\frac{1}{\alpha_t^c} \sum_{\substack{n=1 \\ n \neq m}}^{N} \alpha_n^c \frac{\partial \tilde{U}_{jx}(\mathbf{p},\mathbf{p}_n)}{\partial p_i}, \qquad i, j = x, y$$

$$\frac{\partial \tilde{U}_{jy}(\mathbf{p},\mathbf{p}_m)}{\partial p_i} = -\frac{1}{\beta_t^c} \sum_{\substack{n=1 \\ n \neq m}}^{N} \beta_n^c \frac{\partial \tilde{U}_{jy}(\mathbf{p},\mathbf{p}_n)}{\partial p_i}, \qquad i, j = x, y$$

Now we have calculated all elements of the matrix and α_n and β_n .

$$\begin{bmatrix} \tilde{U}_{xx} & \tilde{U}_{xy} \\ \tilde{U}_{yx} & \tilde{U}_{yy} \end{bmatrix} \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix} = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

Example 1

$$\nu = 0.25, \quad E = 1\,\mathrm{N/m^2}$$

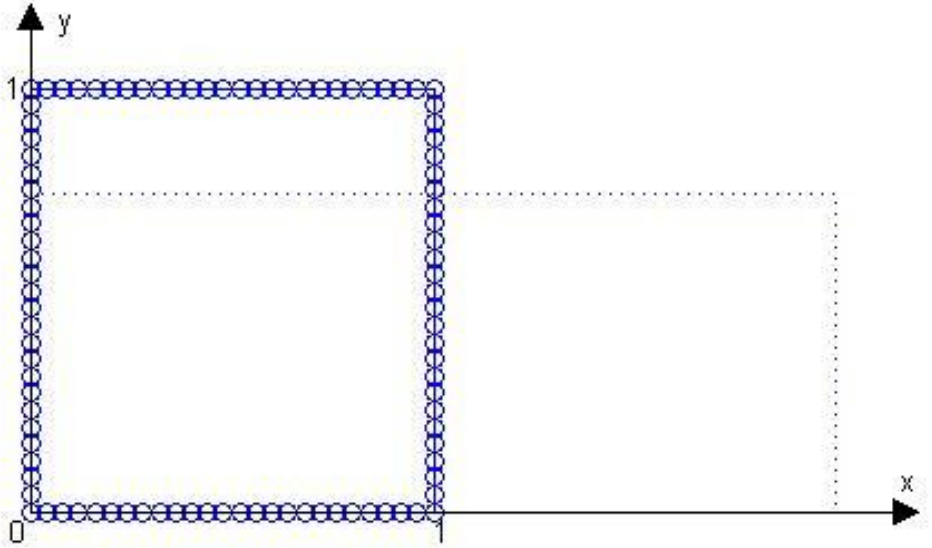
Boundary conditions:

$$\bar{u}_x = p_x \quad \bar{u}_y = -\frac{1}{4}p_y$$

Analytical solution:

$$u_x = p_x$$

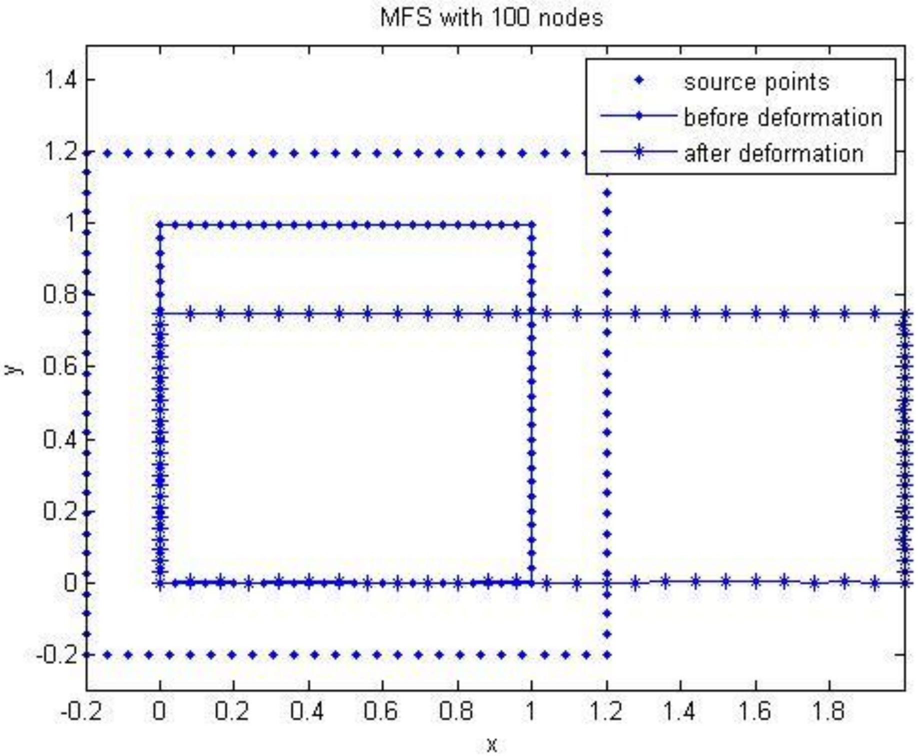
$$u_y = -\frac{1}{4}p_y$$



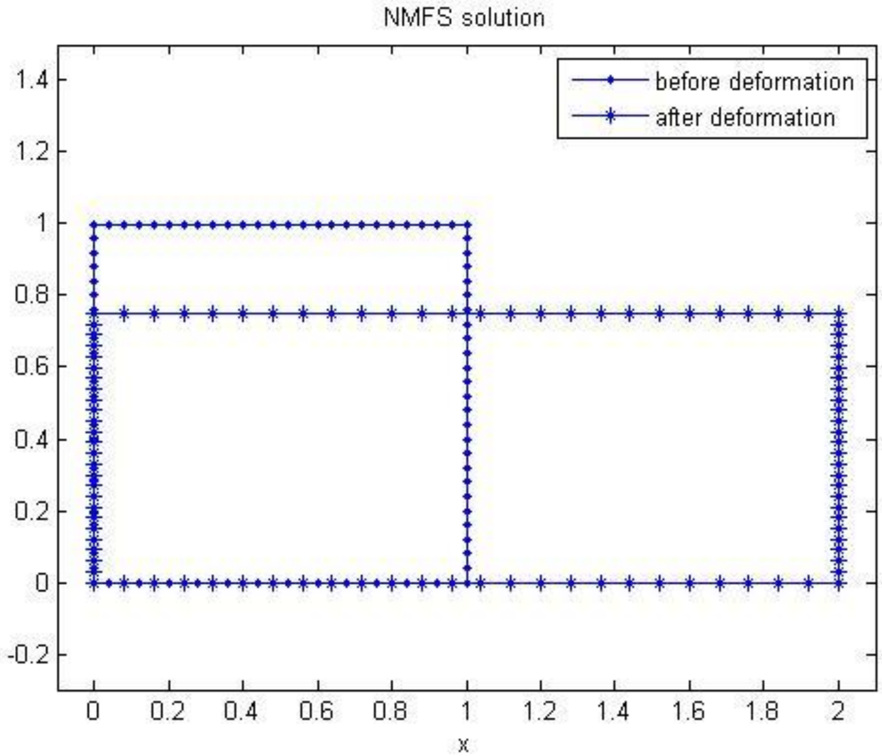
Example 1

$R' = 5d$

$R = d / 3$



MFS



NMFS

Example 1

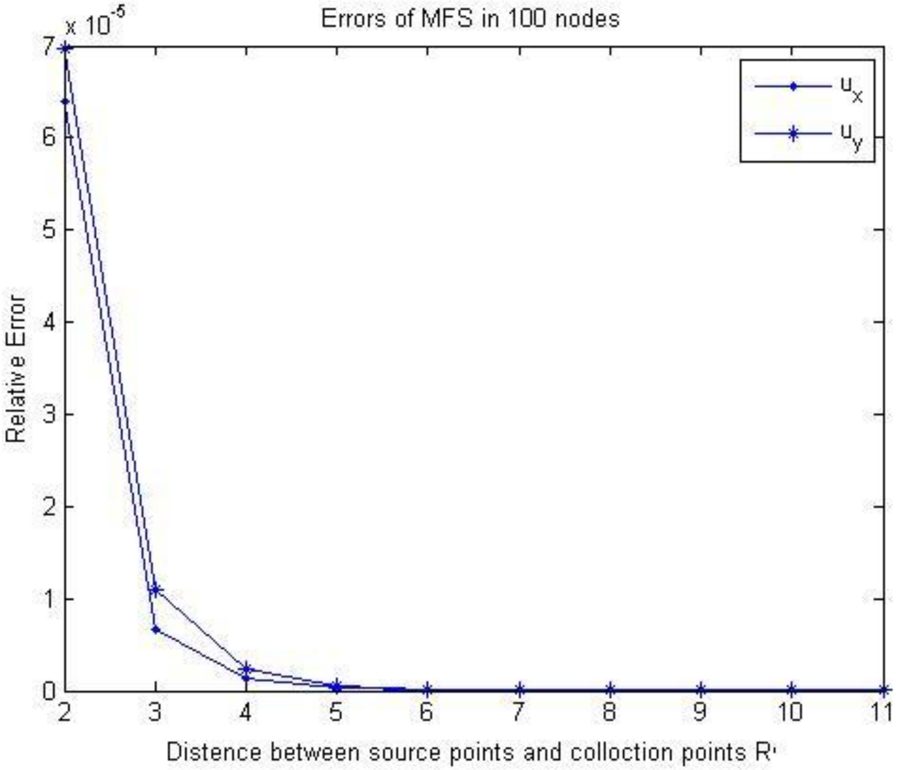
The number of boundary nodes used is 100. A number M=19 of field points are selected inside the domain along the line y=0.5 with $0.5 \le x \le 0.95$. The solutions for $R = d / N (N = 2, 3, \cdots, 11)$ at these field points are computed with NMFS and compared with the analytical solutions. The relative errors of the numerical solutions are defined as .

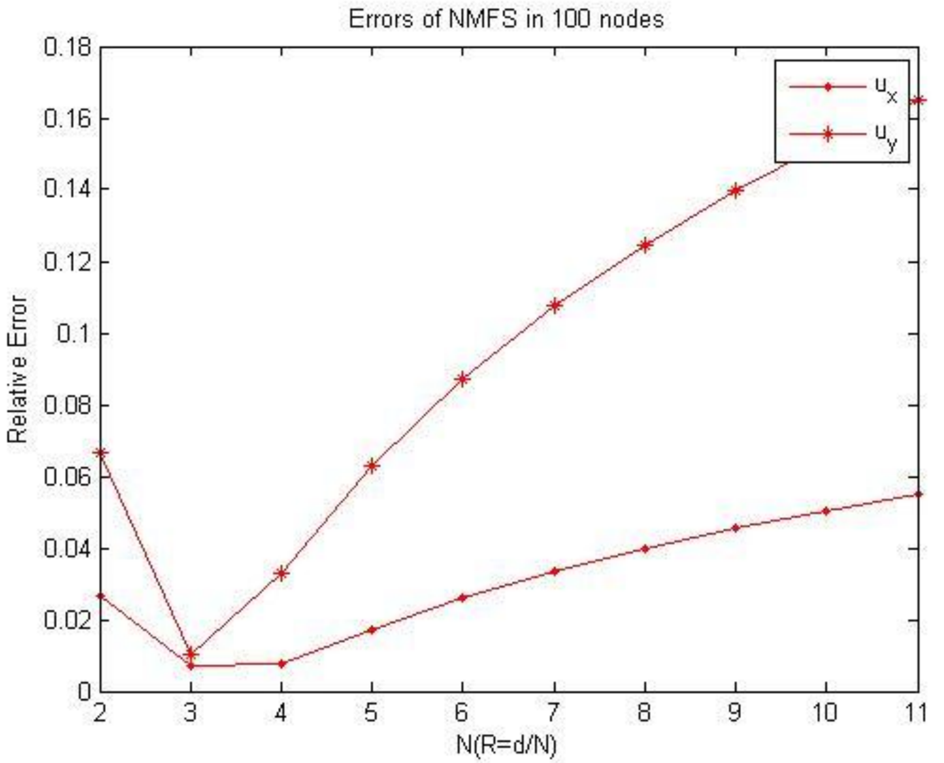
$$Err = \left[\frac{1}{M} \sum_{k=1}^{M} ((u_i - u_i^a) / u_i^a)^2 \right]^{1/2}$$

The solutions for the distance $R' = N \cdot d (N = 2, 3, \cdots, 11)$ of the fictitious boundary from the true boundary are also computed with MFS and compared with the analytical solutions.

Example 1

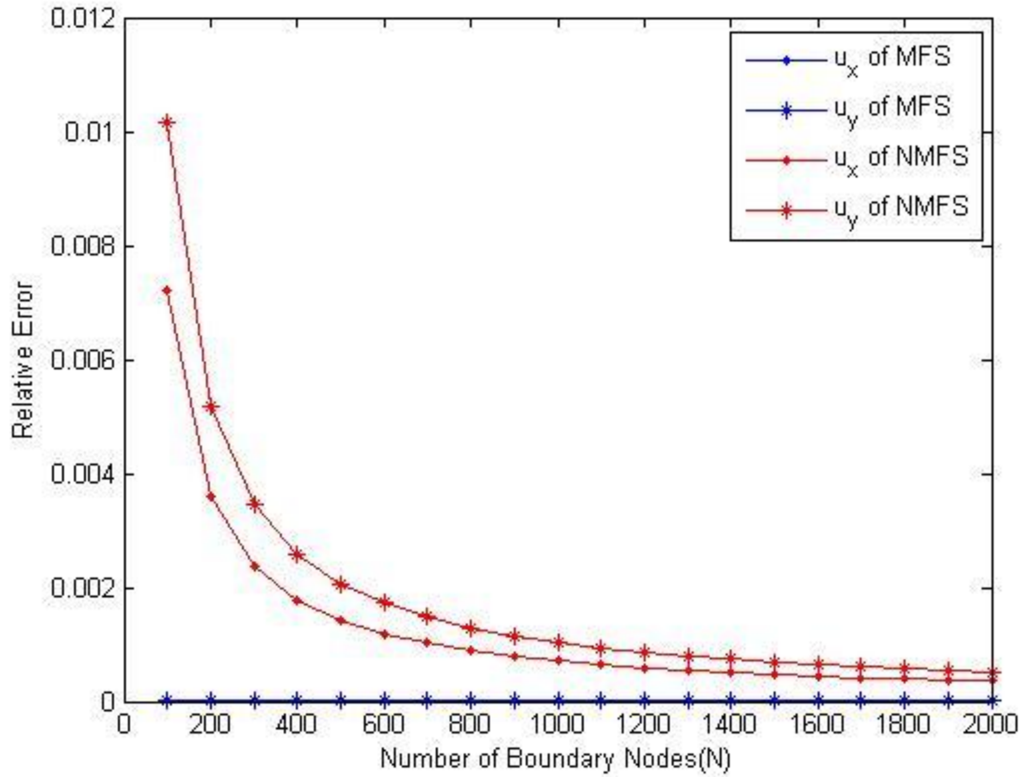
With $R' = 5d$, $R = d/3$



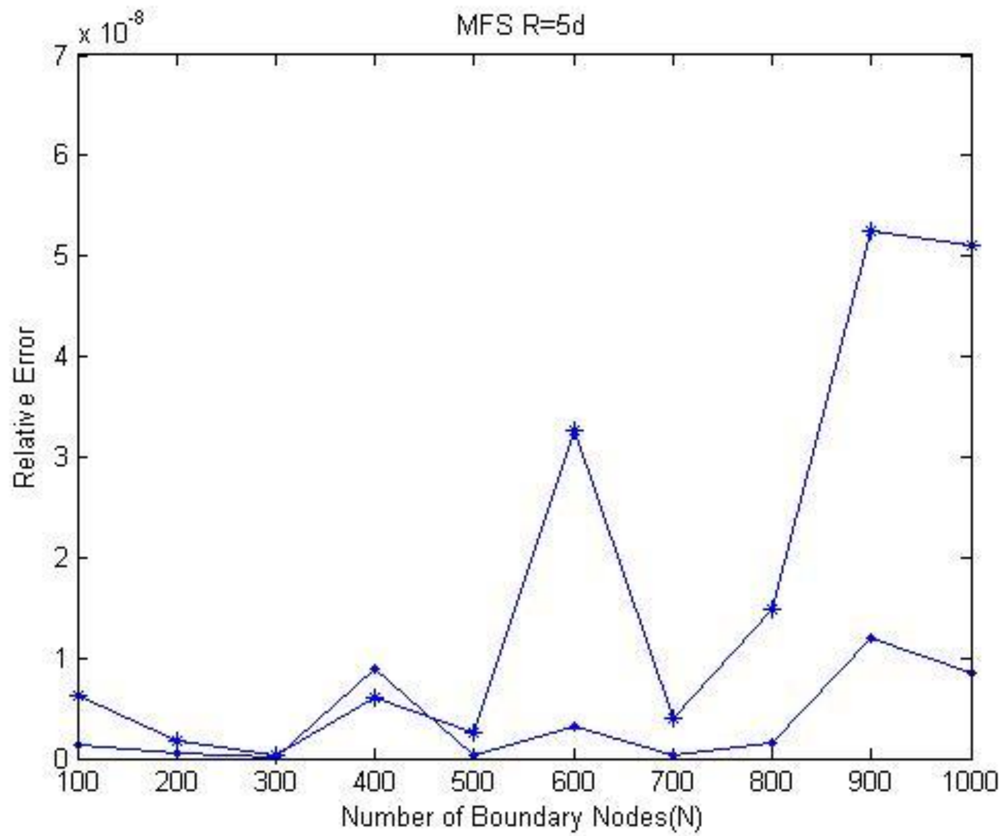


Example 1

The number of boundary nodes used is from 100 to 2000. $R=d/3$ $R'=5d$. We also use the 19 points we used just now to compute. The solutions are compared with the analytical solutions. We know the more nodes we get, the relative errors will be smaller with the NMFS. In this case errors will be very small with the MFS.



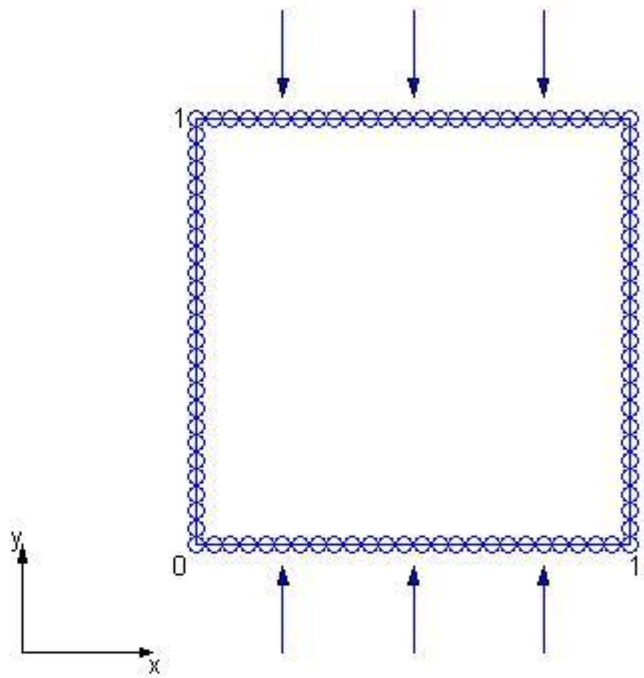
Example 1



The matrix becomes ill conditioned in MFS when the number of points is growing

Example 2

$$\nu = 0.35678 \quad E = 97362.21 \times 10^{6}\,\mathrm{N/m^2}$$



Example 2

Boundary condition:

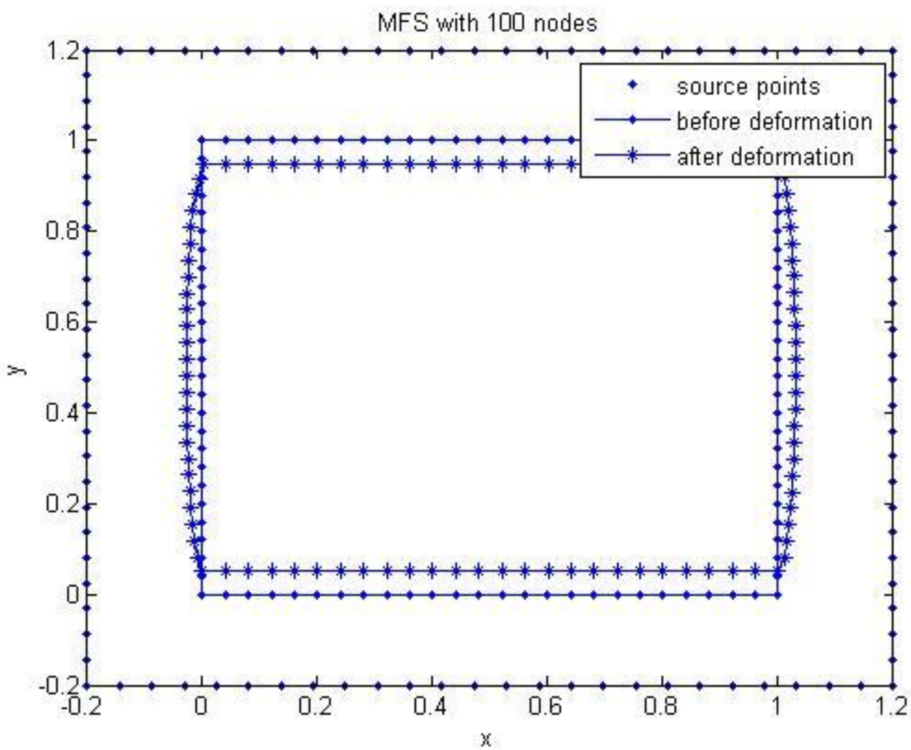
$$y = 0: \bar{u}_x = 0 \quad \bar{u}_y = 0.05$$

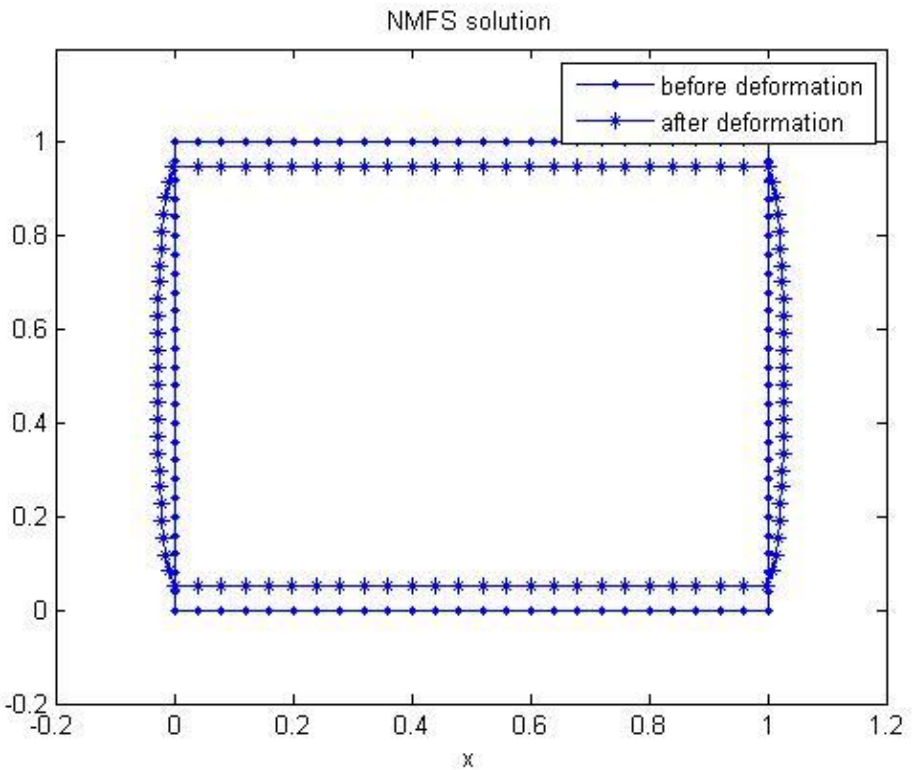
$$y = 1: \bar{u}_x = 0 \quad \bar{u}_y = -0.05$$

$$x = 0 \text{ or } 1: \bar{t}_x = 0 \quad \bar{t}_y = 0$$

$$R' = 5d$$

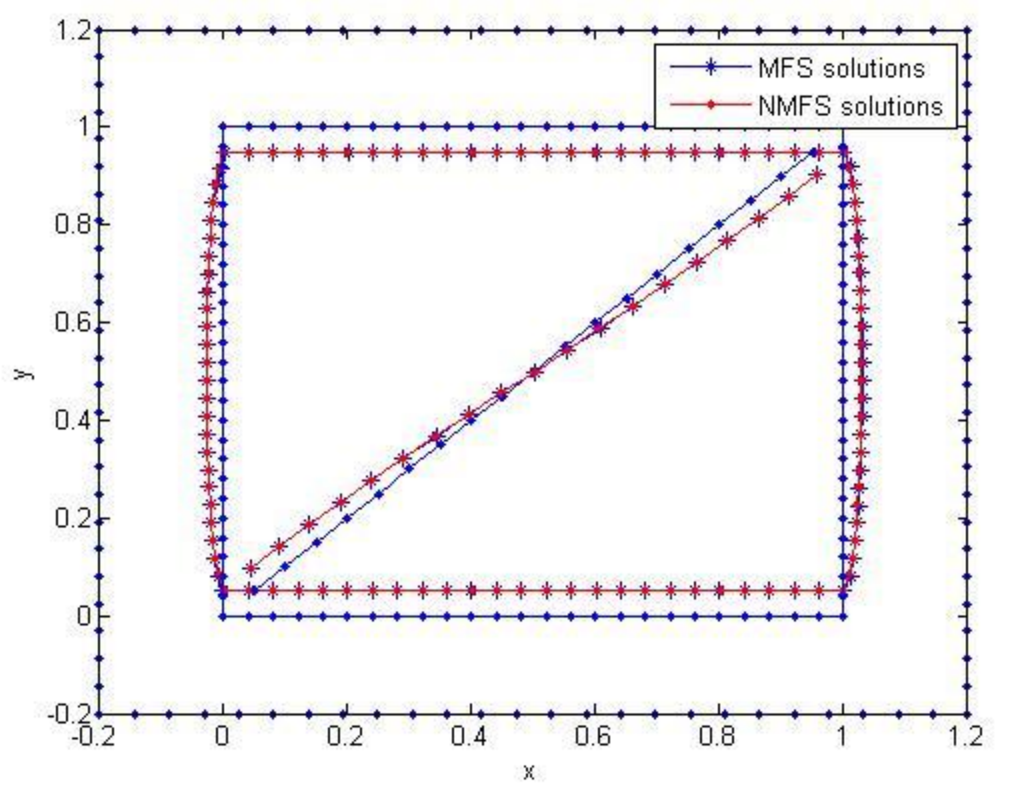
$$R = d/3$$





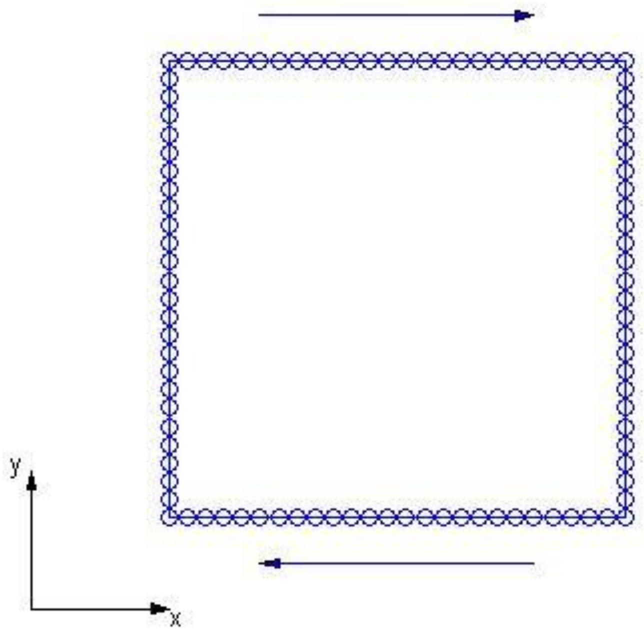
Example 2

We put MFS and NMFS solutions together and use red color to express the NMFS results, blue color to denote the MFS results. And a number of M=19 field points are selected inside the domain along the line $0.05 \le x = y \le 0.095$ we can see that the NMFS results and the MFS results almost coincide and both on the boundary and in the square. So both methods can be used.



Example 3: shear deformation

$$\nu = 0.35678 \quad E = 97362.21 \times 10^6 \, \mathrm{N/m^2}$$



Material properties used are for steel.

Example 3: shear deformation

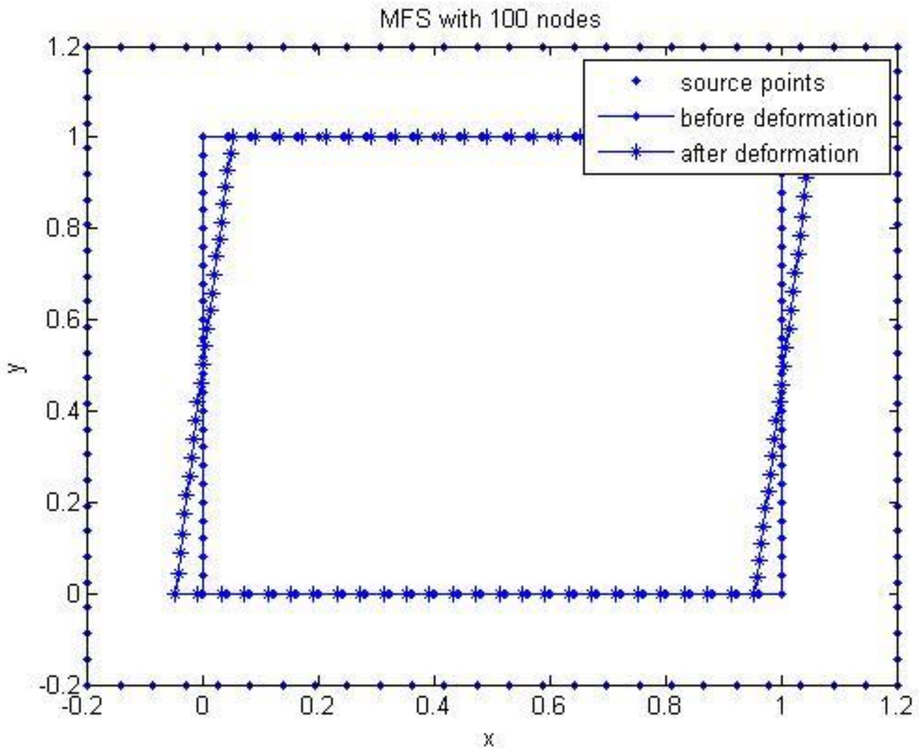
Boundary condition :

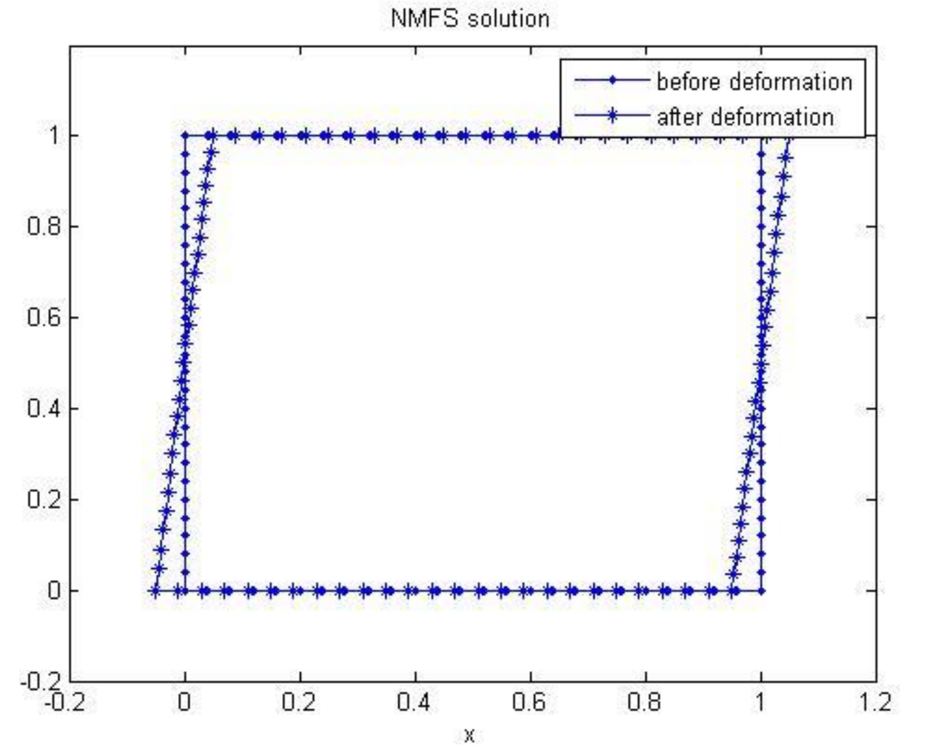
$$y=0:\bar{u}_x=0 \quad \bar{u}_y=0.05$$

$$y=1:\bar{u}_x=0 \quad \bar{u}_y=-0.05$$

$$x=0 or 1:\bar{t}_x=0 \quad \bar{t}_y=0$$

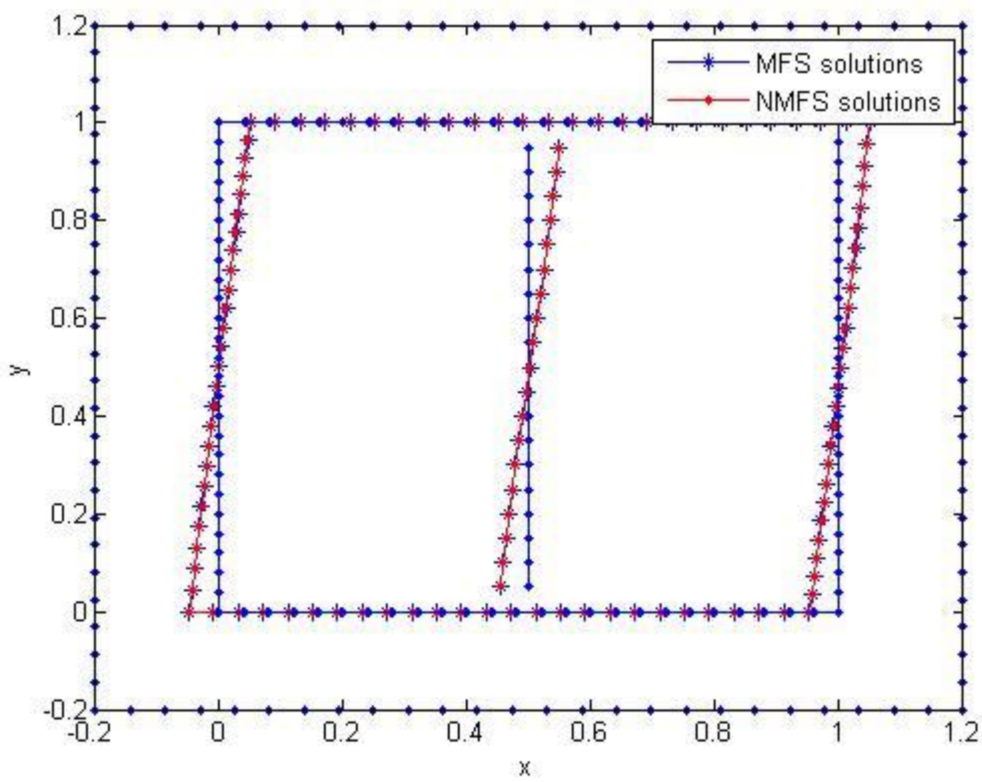
$$R'=5d \qquad\qquad R=d/3$$





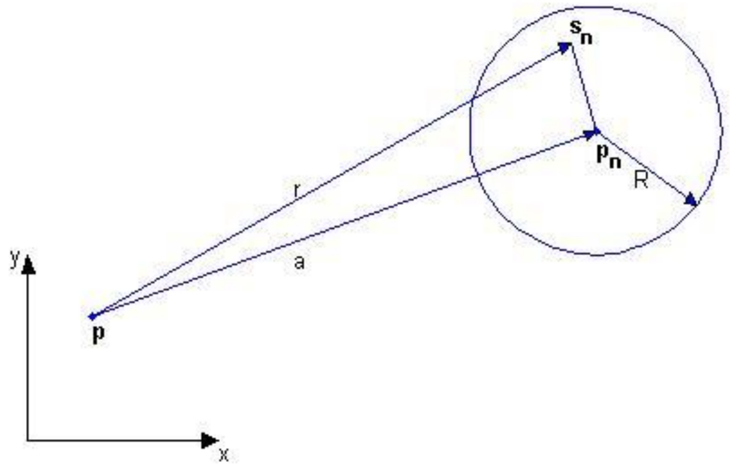
Example 3: shear deformation

We put MFS and NMFS solutions together and use red color to express the NMFS results, blue color to denote the MFS results. And a number of M=19 field points are selected in side the domain along the line x=0.5 with $0.05 \leq y \leq 0.095$. We can see that the NMFS resuilts and MFS resuilts are almost the same. So both methods can be used.

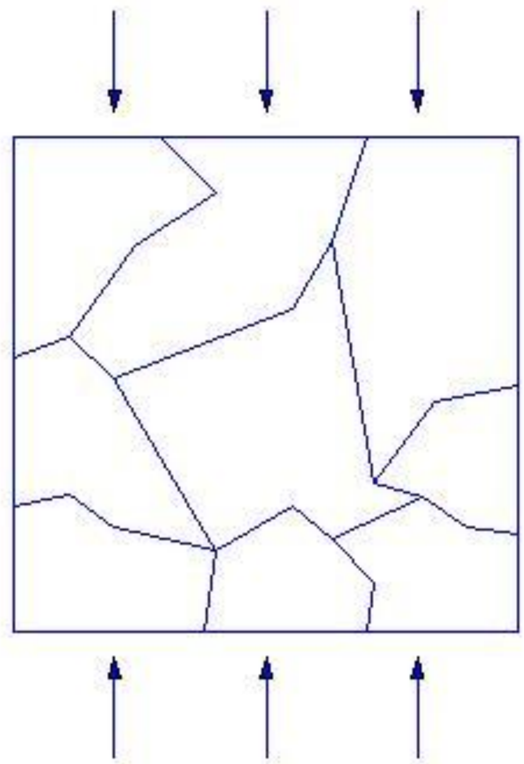


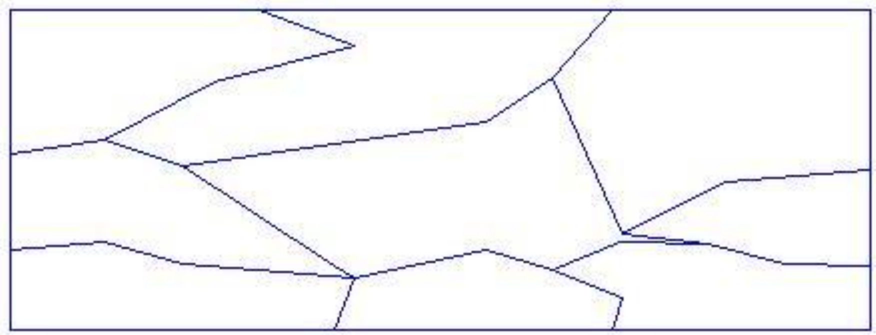
Conclusions

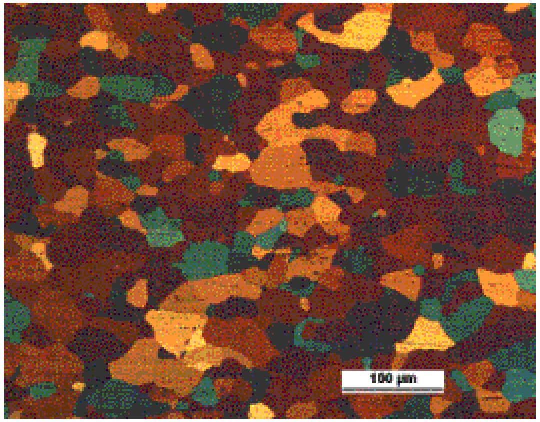
- Solving boundary-value problems with the nodes on the boundary only
- First time used for elasticity problems
- Based on: modified MFS
- Integrated first
- Determining the diagonal elements: Šarler, 2008, 2009

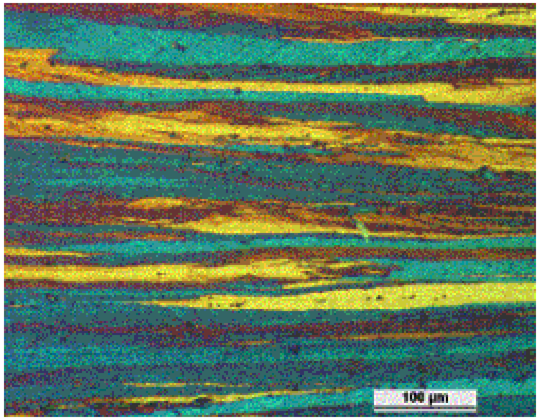


Future Work









References

[1] Fairweather G.; Karageorghis A. The method of fundamental solutions for elliptic boundary value problems. Adv. Comput. Math. 1998, 9: 69–95.

[2] Golberg M.A.; Chen C.S. Discrete Projection Methods for Integral Equations, Computational Mechanics Publications, Southampton, 1997.

[3] Golberg M.A., Chen C.S. The method of fundamental solutions for potential, Helmholtz and diffusion problems, in: Boundary Integral Methods: Numerical and Mathematical Aspects, Computational Engineering, vol. 1, WIT Press/Computational Mechanics Publications, Boston, MA, 1999, pp. 103–176.

[4] Kołodziej J.A. Review of applications of the boundary collocation methods in mechanics of continuous media. Solid Mech. Arch. 1987, 12: 187–231.

[5] Kołodziej J.A. Zastosowanie metody kollokacji brzegowej w zagadnieniach mechaniki [Applications of the Boundary Collocation Method in Applied Mechanics], Wydaw. Politechniki Poznánskiej, Poznán, 2001 (in Polish).

References

[6] Kupradze V. D. Aleksidze M. A. The method of functional equations for the approximate solution of certain boundary value problems, Comput. Methods Math. Phys. 1964, 4: 82–126.

[7] Kupradze V.D. On a method of solving approximately the limiting problems of mathematical physics. Ž. Vyčisl. Mat. i Mat. Fiz. 1964, 4: 1118–1121.

[8] Smyrlis Y.-S. Applicability and applications of the method of fundamental solutions, Math. Comp. (2006), 78: 1399-1434.

[9] Berger J.R., Karageorghis A. The method of fundamental solutions for layered elastic materials. Eng. Anal. Bound. Elem. 2001, 25: 877–886.

[10] Maharejin E. An extension of the superposition method for plane anisotropic elastic bodies. Comput. & Structures 1985, 21: 953–958.

[11] Marin L.; Lesnic D. The method of fundamental solutions for the Cauchy problem in two-dimensional linear elasticity. International J. Solids Structure, 2004, 41: 3425–3438.

[12] Karageorghis A.; Fairweather G. The method of fundamental solutions for axisymmetric elasticity problems. Comput. Mech. 2000 25: 524–532.

References

[13] Redekop D.; Thompson J.C. Use of fundamental solutions in the collocation method in axisymmetric elastostatics, Comput. & Structures, 1983, 17: 485–490.

[14] Berger J.R.; Karageorghis A.; Martin P.A. Stress intensity factor computation using the method of fundamental solutions: mixed-mode problems. Internat. J. Numer. Methods Eng. 2005, 1: 1–13.

[15] Karageorghis A.; Poullikkas A.; Berger J. R. Stress intensity factor computation using the method of fundamental solutions. Comput. Mech. 2006, 37: 445–454.

[16] Aleksidze M. A. Fundamentalnye funktsii v priblizhennykh resheniyakh granichnykh zadach [Fundamental functions in approximate solutions of boundary value problems], Spravochnaya Matematicheskaya Biblioteka [Mathematical Reference Library], “Nauka”, Moscow, 1991. (in Russian); (with an English summary.).

References

[17] Kupradze V.D.; Gegelia T.G.; Basheleshvili M.O.; Burchuladze T.V. Trekhmernye zadachi matematicheskoi teorii uprugosti i termouprugosti [Three-dimensional problems in the mathematical theory of elasticity and thermoelasticity.], Izdat. "Nauka", Moscow, 1976 (in Russian).

[18] Burgess G.; Maharejin E. A comparison of the boundary element and superposition methods. Comput. & Structures. 1984, 19: 697–705.

[19] Fenner R.T. A force supersition approach to plane elastic stress and strain analysis. J. Strain Anal. 2001, 36: 517–529.

[20] Patterson C.; Sheikh M.A. On the use of fundamental solutions in Trefftz method for potential and elasticity problems, in: C.A. Brebbia (Ed.), Boundary Element Methods in Engineering, Proceedings of the Fourth International Seminar on Boundary Element Methods, Springer, New York. 1982, 43–54.

[21] Poullikkas A.; Karageorghis A.; Georgiou G. The method of fundamental solutions for three-dimensional elastostatics problems, Comput. & Structures, 2002, 80: 365-370.

[22] Raamachandran J.; Rajamohan C. Analysis of composite using charge simulation method, Eng. Anal. Bound. Elem. 1996, 18: 131–135.

References

[23] Redekop D. Fundamental solutions for the collocation method in planar elastostatics, Appl. Math. Modelling, 1982, 6: 390–393.

[24] Redekop D.; Cheung R.S.W. Fundamental solutions for the collocation method in three-dimensional elastostatics, Comput. & Structures, 1987; 26: 703–707.

[25] Young D. L.; Chen K. H.; Lee C.W. Novel meshless method for solving the potential problems with arbitrary domain. Journal of Computational Physics 2005, 209: 290-321.

[26] Young D. L.; Chen K. H.; Chen J. T.; Kao J. H. A modified method of fundamental solutions with source on the boundary for solving Laplace equations with circular and arbitrary domains. CMES: Computer Modeling in Engineering & Sciences 2007, 19(3): 197–221.

[27] Chen K. H.; Kao J. H.; Chen J. T.; Young D. L.; Lu M. C. Regularized meshless method for multiply-connected-domain Laplace problems. Engineering Analysis with Boundary Elements, 2006, 30: 882–96.

References

[28] Šarler B. Solution of potential flow problems by the modified method of fundamental solutions: formulations with the single layer and the double layer fundamental solutions. Engineering Analysis with Boundary Elements, 2009, 33(12): 1374–82.

[29] Chen C. S.; Karageorghis A.; Smyrlis Y. S. The method of fundamental Solutions – a meshless method. Dynamic Publishers,Inc. 2008.

[30] Chen W.; Wang F. Z. A method of fundamental solutions without fictitious boundary. Engineering Analysis with Boundary Elements, 2010, 34(5): 530–532.

[31] Liu. Y. J. A new boundary meshfree method with distributed sources. Engineering Analysis with Boundary Elements 2010, 34: 914–919.

[32] Beskos D. E. Boundary element methods in mechanics. Elasevier Science Publishers B. V., 1987.

[34] Zieniuk E.; Boltuc A. Non-element method of solving 2D boundary problems defined on polygonal domains modeled by Navier equation. International Journal of Solids and Structures, 2006, 43: 7939–7958

Thank you!